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CHAPTER

12 Sequences and Series

A sky diver falls 10 meters during the first second, 20 meters during the second second, 30 meters during the third second, and so on. How many meters will the diver fall during the eleventh second?

12–1 ■ Sequences

A sequence

In chapter 10, we discussed the concept of a function. Now we wish to consider a very special function—a function whose domain is the set of positive integers. Such a function is called a **sequence.** There are two kinds of sequences, finite and infinite.

Definition of a sequence

An **infinite sequence** is a function whose domain is the set of positive integers $\{1,2,3, \cdots\}$. A **finite sequence** is a sequence whose domain is the first n positive integers.

Examples of sequences are

$$2,5,8,11,14, \cdots \longleftarrow$$ Three dots indicate an *infinite* sequence (never ends)

where each term of the sequence is found by adding 3 to the preceding term, and

$$10,20,30,40,50. \longleftarrow$$ Period indicates a *finite* sequence (ends here)

where each term of the sequence is found by adding 10 to the preceding term.

Sequences play an important role in the fields of science, finance, and mathematics. For example, the periodic amount of money in a savings account that is compounded at regular intervals is a special kind of sequence.

The numbers that make up the sequence are called the *terms* of the sequence. A general sequence is written

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

where

$$\begin{aligned} a_1 &= \text{first term} \\ a_2 &= \text{second term} \\ a_3 &= \text{third term} \\ &\vdots \\ a_n &= n\text{th term (often called the } \textit{general term} \text{ of the sequence).} \end{aligned}$$

Note The *subscript* in each term represents the term number.

When we think of each term of a sequence as a *function of n,* where n is the term number and *n is a natural number,* we write

$$a_n = f(n)$$

■ Example 12–1 A

1. Given $a_n = f(n) = 3n + 1$, write the terms as an infinite sequence.

$$\begin{aligned} a_1 &= f(1) = 3(1) + 1 = 4 && \text{Replace } n \text{ with 1} \\ a_2 &= f(2) = 3(2) + 1 = 7 && \text{Replace } n \text{ with 2} \\ a_3 &= f(3) = 3(3) + 1 = 10 && \text{Replace } n \text{ with 3} \\ &\vdots \\ a_9 &= f(9) = 3(9) + 1 = 28 && \text{Replace } n \text{ with 9} \\ &\vdots \end{aligned}$$

The infinite sequence is $4, 7, 10, \cdots, 28, \cdots$

2. Given $a_n = f(n) = \dfrac{n-2}{3}$, find the first five terms of the finite sequence.

$$\begin{aligned} a_1 &= f(1) = \frac{(1)-2}{3} = -\frac{1}{3} && \text{Replace } n \text{ with 1} \\ a_2 &= f(2) = \frac{(2)-2}{3} = 0 && \text{Replace } n \text{ with 2} \\ a_3 &= f(3) = \frac{(3)-2}{3} = \frac{1}{3} && \text{Replace } n \text{ with 3} \\ a_4 &= f(4) = \frac{(4)-2}{3} = \frac{2}{3} && \text{Replace } n \text{ with 4} \\ a_5 &= f(5) = \frac{(5)-2}{3} = 1 && \text{Replace } n \text{ with 5} \end{aligned}$$

The finite sequence is $-\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$ and contains five terms.

3. Given $a_n = 5n - 7$, find a_7.
Since we want a_7, then $n = 7$ and

$$\begin{aligned} a_7 &= 5(7) - 7 && \text{Replace } n \text{ with 7} \\ &= 35 - 7 \\ &= 28 \end{aligned}$$

▶ ***Quick check*** Given $a_n = 2n + 5$, write the first five terms of the infinite sequence.

Finding the general term

On occasion, we may be given the first few terms of a sequence and asked to find an expression for the general term, a_n. There are no rules for finding this from the first few terms of the sequence. We usually do this by inspection or trial and error. However we should always be aware of one clue. Consider the sequences

1. $a_n = 5n + 2$, whose terms are $7,12,17,22,27, \cdots$; each term of the sequence differs by 5, and the coefficient of n is 5; and
2. $a_n = (-1)^n(n + 7)$, whose terms are $-8,9,-10,11,-12, \cdots$; each term alternates in sign, caused by the factor -1, to some positive integer power; the numerical value of each term differs by 1 and the coefficient of n is 1.

Example 12–1 B

Find an expression for the general term of the given sequence.

1. $5,7,9,11, \cdots$

The difference between each term is 2, so we conclude that $2n$ must be part of the general term. Now if we consider the first term a_1, let $n = 1$ and ask ourselves, $2(1) +$ (what number?) $= 5$. Since $2(1) + 3 = 5$, we think $a_n = 2n + 3$. Check this with the succeeding terms.

$a_2 = 2(2) + 3 = 4 + 3 = 7$ (True)
$a_3 = 2(3) + 3 = 6 + 3 = 9$ (True)

Therefore $a_n = 2n + 3$.

2. $\frac{3}{4}, -\frac{9}{8}, \frac{27}{16}, -\frac{81}{32}, \cdots$

First, the signs alternate and the first term a_1 is positive. We must then have a factor of -1 to an even power when $n = 1$. Therefore one factor of the nth term could be $(-1)^{n+1}$ since $(-1)^{1+1} = (-1)^2 = 1$. Inspection shows us that the numerator of each term is a power of 3, starting with 3^1. Inspecting the denominator, we see that each denominator is a power of 2. When $n = 1$, the denominator of the first term could be

$$2^{n+1} = 2^{1+1} = 2^2 = 4$$

The denominator of the second term, when $n = 2$, is then

$$2^{n+1} = 2^{2+1} = 2^3 = 8$$

and so on. Therefore we conclude that the numerator is 3^n and the denominator is 2^{n+1}. Thus

$$a_n = (-1)^{n+1}\frac{3^n}{2^{n+1}}$$

Note $(-1)^{n-1}$ could be used instead of $(-1)^{n+1}$.

▶ ***Quick check*** Find an expression for the general term of the sequence $4,9,14,19, \cdots$

Mastery points

Can you

- Find the terms of a sequence, given the general term?
- Find any given term of the sequence?
- Define an expression for the general term, given a sequence?

Exercise 12–1

Write the first five terms of the sequence whose general term a_n is given. See example 12–1 A.

Example $a_n = 2n + 5$

Solution
$a_1 = 2(1) + 5 = 7$ Replace n with 1
$a_2 = 2(2) + 5 = 9$ Replace n with 2
$a_3 = 2(3) + 5 = 11$ Replace n with 3
$a_4 = 2(4) + 5 = 13$ Replace n with 4
$a_5 = 2(5) + 5 = 15$ Replace n with 5

The first five terms of the sequence are 7,9,11,13, and 15.

1. $a_n = 4n + 3$ **2.** $a_n = 5n - 4$ **3.** $a_n = \dfrac{2}{3n}$ **4.** $a_n = \dfrac{5}{2n + 1}$

5. $a_n = \dfrac{n + 5}{3n - 4}$ **6.** $a_n = \dfrac{4n}{5n + 2}$ **7.** $a_n = \dfrac{2^n}{5n}$ **8.** $a_n = \dfrac{3^{n+1}}{n + 2}$

9. $a_n = (-1)^n(6n - 5)$ **10.** $a_n = (-1)^{n+1}(n + 1)^2$ **11.** $a_n = (-1)^{n-1}\dfrac{3^n}{2^n + 1}$

12. $a_n = (-1)^n\dfrac{3}{4^n + 2}$ **13.** $a_n = (-1)^{2n}$ **14.** $a_n = (-1)^{3n-1}$

Find the indicated term of the given sequence. See example 12–1 A–3.

15. $a_n = 3n + 5$, find a_9. **16.** $a_n = 7 - 2n$, find a_{11}. **17.** $a_n = \dfrac{1}{3n}$, find a_{37}.

18. $a_n = \dfrac{3}{5n}$, find a_{51}. **19.** $a_n = \dfrac{4n - 3}{2n + 7}$, find a_{17}. **20.** $a_n = \dfrac{9 - 5n}{-6 - 3n}$, find a_{12}.

21. $a_n = (-1)^n(6n + 5)$, find a_{14}. **22.** $a_n = (-1)^{n-1}(5n - 6)$, find a_{15}.

23. $a_n = (-1)^{n+2}n(n + 6)$, find a_{13}. **24.** $a_n = 2n^2(3n - 1)$, find a_8.

Given the terms of the following sequences, find an expression for the general term a_n. See example 12–1 B.

Example 4,9,14,19, · · ·

Solution The common difference between each term is 5, $5n$ is part of the general term. Let $n = 1$ and consider

$a_1 = 5(1) + \text{(what number)?} = 4$
$\quad = 5(1) + (-1) = 4$

Thus, we conclude $a_n = 5n - 1$. To check this,

$a_2 = 5(2) - 1 = 10 - 1 = 9$ Second term

25. 6,8,10,12,14, · · ·

26. 1,4,7,10,13, · · ·

27. 2,7,12,17,22, · · ·

28. 1,4,9,16, · · ·

29. 1,8,27,64,125, · · ·

$\boxed{\textbf{30.}}$ $\frac{1}{3},\frac{1}{9},\frac{1}{27},\frac{1}{81},\frac{1}{243},\cdots$

31. $1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},\cdots$

32. $\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},\frac{6}{7},\cdots$

33. $\frac{3}{5},\frac{4}{7},\frac{5}{9},\frac{6}{11},\frac{7}{13},\cdots$

34. $3,\frac{8}{3},\frac{7}{3},2,\frac{5}{3},\cdots$

$\boxed{\textbf{35.}}$ $-6,10,-14,18,-22,\cdots$

36. $-\frac{3}{2},3,-\frac{9}{2},6,-\frac{15}{2},\cdots$

Solve the following word problems.

$\boxed{37.}$ A culture of bacteria triples every hour if the original culture has 1,000 bacteria. How many bacteria were there after (a) 3 hours, (b) 5 hours, (c) n hours?

38. A pendulum swings a distance of 20 inches on its first swing. If each subsequent swing back is three-fourths of the previous swing, how far does it swing on the (a) third swing, (b) seventh swing?

39. A ball is dropped from a height of 10 feet. If the ball rebounds one-half the height of its previous fall, how high does it rebound on the (a) second bounce, (b) sixth bounce, (c) nth bounce?

40. Jim Jarrett gives his son an allowance each month of 10¢ on the first day, 15¢ on the second day, 20¢ on the third day, and so on. Write a sequence for the first ten days of the month. Write an expression for the amount received on the nth day of the month. How much does he receive on the thirtieth day?

41. Steve Navarro begins a new job at a starting yearly salary of $16,000, with a promise of a salary increase of $1,500 per year for the first 6 years. (a) Write a sequence showing his salary during the first 6 years. (b) Write the general term for the sequence. (c) If this yearly pay increase were to continue beyond 6 years, what would his salary be after 20 years?

Review exercises

1. Given $f(x) = 5x + 4$, find (a) $f(-2)$, (b) $f(0)$, (c) $f(2)$. See section 10–2.

2. Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & -3 & 0 \end{vmatrix}$. See section 8–4.

3. Find the solution set of the system of equations

$$2x - 3y = 4$$
$$x + 2y = 0$$

using determinants. See section 8–5.

Find the standard equation of the following lines. See section 7–3.

4. Vertical line passing through $(-3,4)$

5. Parallel to $x + 3y = 2$ and passing through $(-1,-1)$

6. Find the solution set of the exponential equation $3^{2x+3} = 9$. See section 11–1.


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12–2 ■ Series

A series

Associated with any sequence is the sum of the terms of the sequence, called a **series.**

Definition of a series

A **series** is the indicated sum of the terms in a sequence.

If the sequence is

1. finite, we have a finite series, such as

$$a_1 + a_2 + a_3 + a_4$$

2. infinite, we have an infinite series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

To illustrate, given the sequence whose general term is $a_n = 2n + 1$, the *sequence* is

$$3,5,7,9, \cdots$$

and the *infinite series* of this sequence is

$$3 + 5 + 7 + 9 + \cdots + (2n + 1) + \cdots$$

Definition of a partial sum

A **partial sum** of a series, denoted by S_n, is the sum of a finite number of consecutive terms of the series starting with a_1.

Thus, $S_1 = a_1$ First partial sum

$S_2 = a_1 + a_2$ Second partial sum

$S_3 = a_1 + a_2 + a_3$ Third partial sum

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ nth partial sum

■ Example 12–2 A

Find S_1, S_2, S_3, and S_4 for the following sequences.

1. $a_n = 5n + 4$

Now $a_1 = 5(1) + 4 = 9$
$a_2 = 5(2) + 4 = 14$
$a_3 = 5(3) + 4 = 19$
$a_4 = 5(4) + 4 = 24$

then $S_1 = a_1 = 9$
$S_2 = a_1 + a_2 = 9 + 14 = 23$
$S_3 = a_1 + a_2 + a_3 = 9 + 14 + 19 = 42$
$S_4 = a_1 + a_2 + a_3 + a_4 = 9 + 14 + 19 + 24 = 66$

2. $a_n = (-1)(n - 2)$

Now $a_1 = (-1)(1 - 2) = 1$
$a_2 = (-1)(2 - 2) = 0$
$a_3 = (-1)(3 - 2) = -1$
$a_4 = (-1)(4 - 2) = -2$

then $S_1 = 1$
$S_2 = 1 + 0 = 1$
$S_3 = 1 + 0 + (-1) = 0$
$S_4 = 1 + 0 + (-1) + (-2) = -2$

▶ ***Quick check*** Given $a_n = 2n - 7$, find S_3. ■

Summation notation

A compact way of representing a sum when the general term is known is by **sigma** (or **summation**) **notation.** To do this, we use the Greek letter sigma, Σ, in conjunction with the general term of the related sequence. To illustrate, consider the sequence

$$3,7,11,15,19,23,27, \cdots$$

We can determine that the general term of this sequence is $4n - 1$. Suppose we want the sum of the first seven terms of the sequence, called the **partial sum.** We want

$$\sum_{i=1}^{7} (4i - 1) = 3 + 7 + 11 + 15 + 19 + 23 + 27 = 105$$

where the expression for the general term, $4n - 1$, becomes $4i - 1$ when n is replaced by i. We call the letter i, as used in this situation, the **index of summation.** Other letters often used for this purpose are j and k.

Note This use of i has no connection with its use in our work with complex numbers.

We read the expression $\sum_{i=1}^{7} (4i - 1)$ "the summation as i goes from 1 to 7 of $4i - 1$." The first and last integers used to replace the index of summation, in this case 1 and 7, are called the **lower and upper limits of summation,** respectively.

■ **Example 12–2 B**

Expand the following indicated partial sums. Find the indicated sum.

1. $\sum_{j=1}^{4} (5j + 2)$

Now the index of summation is j, and we successively replace j by the integers 1, 2, 3, and 4. Thus

$$\begin{aligned}\sum_{j=1}^{4} (5j + 2) &= [5(1) + 2] + [5(2) + 2] + [5(3) + 2] + [5(4) + 2] \\ &= 7 + 12 + 17 + 22 = 58\end{aligned}$$

2. $\sum_{k=1}^{3} (-1)^k(2k + 5)$

We successively replace the index of summation k by the integers 1, 2, and 3. Therefore

$$\begin{aligned}\sum_{k=1}^{3} (-1)^k(2k + 5) &= (-1)^1 [2(1) + 5] + (-1)^2 [2(2) + 5] \\ &\quad + (-1)^3 [2(3) + 5] \\ &= (-1)(7) + 1(9) + (-1)(11) \\ &= -7 + 9 - 11 = -9\end{aligned}$$

▶ ***Quick check*** Expand the partial sum $\sum_{i=1}^{5} (2i - 7)$. Find the indicated sum. ■

We should note that there is nothing unique about the way the above-indicated sums have been stated using sigma notation.

Reversing the procedure, it is sometimes desirable to express a sum in the compact summation notation form.

■ Example 12–2 C

Write the following partial sums in sigma notation.

1. $-5 + 25 - 125 + 625$

There are four terms, so we can use the limits of summation 1 and 4. Since the operations alternate with a negative first term, the general term will contain -1 to an odd power. Use $(-1)^j$ for the first factor of the general term. By inspection, the numerical value of the terms of the series are powers of 5. Thus

$$-5 + 25 - 125 + 625 = \sum_{j=1}^{4} (-1)^j 5^j$$

2. $\frac{2}{3} + \frac{4}{7} + \frac{6}{11} + \frac{8}{15} + \frac{10}{19}$

The limits of summation can be 1 and 5, since there are five terms. Inspection tells us the general term of the numerator is $2n$ and of the denominator is $4n - 1$. Then

$$a_n = \frac{2n}{4n - 1} \text{ and}$$

$$\frac{2}{3} + \frac{4}{7} + \frac{6}{11} + \frac{8}{15} + \frac{10}{19} = \sum_{k=1}^{5} \frac{2k}{4k - 1}$$

▶ ***Quick check*** Write the partial sum $1 + 3 + 5 + 7$ in sigma notation. ■

Mastery points

Can you

- Find the nth partial sum of an infinite series?
- Expand a partial sum that is written using sigma notation?
- Write a partial sum of a series in sigma notation?

Exercise 12–2

Expand the following indicated partial sums. Find the indicated sum. See example 12–2 A.

Example $a_n = 2n - 7; S_3$

Solution Now $a_1 = 2(1) - 7 = -5$ Replace n with 1
$a_2 = 2(2) - 7 = -3$ Replace n with 2
$a_3 = 2(3) - 7 = -1$ Replace n with 3
since $S_3 = a_1 + a_2 + a_3$
$= (-5) + (-3) + (-1)$ Replace a_1 with -5, a_2 with -3, a_3 with -1
$= -9$

1. $a_n = 4n + 3; S_5$
$\boxed{2.}$ $a_n = 5n - 1; S_4$
3. $a_n = \dfrac{3}{n+2}; S_3$
4. $a_n = \dfrac{4}{2n+3}; S_2$
5. $a_n = (-1)(6n - 1); S_3$
6. $a_n = (-1)(n - 9); S_4$
7. $a_n = \dfrac{2n-1}{4-n}; S_3$
8. $a_n = 2^n + 3; S_3$

See example 12–2 B.

Example $\sum_{i=1}^{5} (2i - 7)$

Solution $\sum_{i=1}^{5} (2i - 7) = [2(1) - 7] + [2(2) - 7] + [2(3) - 7] + [2(4) - 7] + [2(5) - 7]$
$= (-5) + (-3) + (-1) + 1 + 3$
$= -5$

9. $\sum_{j=1}^{5} j^2$
10. $\sum_{k=1}^{4} k^3$
11. $\sum_{i=1}^{6} (2i + 3)$
12. $\sum_{i=1}^{7} (i - 2)$
$\boxed{13.}$ $\sum_{k=1}^{5} k(k - 3)$
14. $\sum_{j=1}^{6} (j^2 + 2)$
15. $\sum_{i=1}^{4} i(2i - 1)$
16. $\sum_{i=1}^{4} (i + 1)(i - 2)$
17. $\sum_{j=1}^{5} (j - 3)(j + 2)$
18. $\sum_{k=1}^{4} k^2(k + 2)$
$\boxed{19.}$ $\sum_{k=1}^{5} \dfrac{1}{k+3}$
20. $\sum_{i=1}^{4} \dfrac{3}{3i-1}$
21. $\sum_{k=1}^{5} \dfrac{2k+1}{k+3}$
22. $\sum_{j=1}^{4} \dfrac{j^2}{3j-2}$
23. $\sum_{i=1}^{5} \dfrac{4}{i^2}$
$\boxed{24.}$ $\sum_{j=1}^{5} (-1)^j \cdot \dfrac{3}{2j}$
25. $\sum_{k=1}^{4} (-1)^{k+1} \cdot \dfrac{1}{3k}$
26. $\sum_{i=1}^{5} (-1)^i i$
27. $\sum_{j=1}^{3} (-1)^{j-1}(j)^j$
28. $\sum_{k=1}^{5} (-1)^{k+1}(k + 1)^k$

Expand and find the following indicated partial sums.

Example $\sum_{i=2}^{5} (3i + 1)$

Solution $\sum_{i=2}^{5} (3i + 1) = [3(2) + 1] + [3(3) + 1] + [3(4) + 1] + [3(5) + 1] = 7 + 10 + 13 + 16 = 46$

29. $\sum_{i=3}^{8} (i + 2)$
30. $\sum_{j=0}^{4} (2j + 1)$
31. $\sum_{k=2}^{5} \frac{1}{k}$
32. $\sum_{i=4}^{9} (i^2 + 3)$
33. $\sum_{j=0}^{5} \frac{2j + 1}{j + 1}$
34. $\sum_{k=2}^{6} (-1)^k$
35. $\sum_{i=3}^{7} (-1)^i(3i - 2)$
36. $\sum_{i=4}^{7} (-1)^i \frac{2}{2i + 3}$

Write the following partial sums in sigma notation. See example 12–2 C.

Example $1 + 3 + 5 + 7$

Solution There are four terms, so the limits of summation can be 1 to 4. Inspection tells us there is a common difference of 2 so $2n$ is part of the general term.

$$\begin{aligned} \text{Now } a_1 &= 2(1) + ? = 1 \\ &= 2(1) + (-1) = 1 \\ &= 2(1) - 1 \end{aligned}$$

The general term is $a_n = 2n - 1$, so

$$1 + 3 + 5 + 7 = \sum_{i=1}^{4} (2i - 1) \qquad \text{Replace } n \text{ with } i$$

37. $1 + 2 + 3 + 4 + 5$
38. $3 + 6 + 9 + 12$
39. $1 + 8 + 27 + 64 + 125 + 216$
40. $5 + 9 + 13 + 17 + 21$
41. $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$
42. $\frac{1}{3} + \frac{3}{5} + \frac{5}{7} + \frac{7}{9} + \frac{9}{11}$
43. $2 + \frac{4}{3} + \frac{6}{9} + \frac{8}{27}$
44. $\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16}$
45. $\frac{5}{4} + \frac{7}{7} + \frac{9}{10} + \frac{11}{13}$
46. $2 - 5 + 8 - 11 + 14$
47. $-2 + 4 - 8 + 16 - 32 + 64$
48. $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6}$

Review exercises

Evaluate the following expressions when $a = 2$, $n = 4$, and $r = 3$. See section 1–5.

1. $a + (n - 1)r$
2. $\frac{n}{2}(a + r)$
3. $\frac{n}{2}[a + (n - 1)r]$
4. ar^{n-1}
5. Simplify the complex rational fraction $\dfrac{\frac{1}{3} - \frac{2}{5}}{1 + \frac{1}{5}}$. See section 4–4.
6. Solve the equation $3^{x-1} = 4$ using common logarithms. Round off to four decimal places. See section 11–6.

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12–3 ■ Arithmetic sequences

An arithmetic sequence

Consider the sequence defined by

$$3,7,11,15,19,\cdots,4n-1,\cdots$$

whose major characteristic is that the difference between any two successive terms is 4. Recall that this determines the coefficient of n in the general term. Such a sequence is called an **arithmetic sequence** (or **arithmetic progression**).

Definition of an arithmetic sequence

An **arithmetic sequence** is a sequence in which each term after the first differs from the preceding term by the same constant number.

We call this constant number the **common difference,** to be denoted by d. Thus in the above sequence, the common difference $d = 4$. That is, for any term of an arithmetic sequence,

$$d = a_{n+1} - a_n$$

where d is the common difference and a_n and a_{n+1} are successive terms of the sequence.

■ **Example 12–3 A**

1. The sequence 7,11,15,19 is arithmetic since

   $11 - 7 = 4, \quad 15 - 11 = 4, \quad 19 - 15 = 4$

   and the common difference $d = 4$.

2. The sequence $-10,-4,2,8,14$ is an arithmetic sequence since

   $$\begin{aligned} -4-(-10) &= -4+10 = 6\\ 2-(-4) &= 2+4 = 6\\ 8-2 &= 6\\ \text{and } 14-8 &= 6 \end{aligned}$$

   The common difference $d = 6$.

3. The sequence 1,3,9,12,36 is not an arithmetic sequence since $3 - 1 = 2$ whereas

   $9 - 3 = 6$

▶ ***Quick check*** Determine if the sequence $8,5,2,-1,\cdots$ is arithmetic. If so, find the common difference d. ■

The general term of an arithmetic sequence

To find an expression for the nth (general) term, a_n, we look at the terms, $a_1, a_2, a_3, \cdots$.

$$a_1, \quad a_2 = a_1 + d, \quad a_3 = a_1 + 2d, \quad a_4 = a_1 + 3d, \quad a_5 = a_1 + 4d$$

This suggests the following:

General term of an arithmetic sequence

The general term of an arithmetic sequence with first term a_1 and common difference d is given by

$$a_n = a_1 + (n - 1)d$$

Example 12–3 B

1. Find the twenty-first term of the arithmetic sequence whose first term $a_1 = 3$ and common difference $d = 4$.

   Using $a_n = a_1 + (n - 1)d$, we want a_{21} when $a_1 = 3$, $d = 4$, and $n = 21$.

   $$\begin{aligned} a_{21} &= (3) + [(21) - 1](4) \quad \text{Replace } a_1 \text{ with 3, } n \text{ with 21, and } d \text{ with 4} \\ &= 3 + (20)4 \\ &= 3 + 80 = 83 \end{aligned}$$

   The twenty-first term of the sequence is 83.

2. Given the arithmetic sequence $5, -1, -7, -13, \cdots$, find a_{20}.

   Since $-1 - 5 = -6$, then $d = -6$. Now $a_1 = 5$ and $n = 20$. Using $a_n = a_1 + (n - 1)d$, we want a_{20}.

   $$\begin{aligned} a_{20} &= (5) + [(20) - 1](-6) \quad \text{Replace } a_1 \text{ with 5, } n \text{ with 20, and } d \text{ with } -6 \\ &= 5 + (19)(-6) \\ &= 5 + (-114) \\ &= -109 \end{aligned}$$

   The twentieth term of the sequence is -109.

▶ ***Quick check*** Find the nineteenth term of the arithmetic sequence whose first term is $a_1 = -2$ and the common difference $d = 3$. ■

Given a finite arithmetic sequence, it is possible to determine the number of terms n in the sequence if we can determine d and a_1.

Example 12–3 C

Find the number of terms in the finite arithmetic sequence $-9, -4, 1, 6, \cdots, 111$.

Now from the first 4 terms of the sequence, we can determine $a_1 = -9$, $d = -4 - (-9) = -4 + 9 = 5$, and $a_n = 111$. We want the value of n.

Replacing a_1 by -9, d by 5, and a_n by 111 in the formula $a_n = a_1 + (n - 1)d$, we get

$$\begin{aligned} (111) &= (-9) + (n - 1)(5) \quad \text{Replace } a_n \text{ with 111, } a_1 \text{ with } -9\text{, and } d \text{ with 5} \\ 111 &= -9 + 5n - 5 \\ 111 &= 5n - 14 \\ 125 &= 5n \\ 25 &= n \end{aligned}$$

Thus the sequence has $n = 25$ terms.

▶ ***Quick check*** Find the number of terms in the finite arithmetic sequence $-5, -1, 3, 7, \cdots, 115$. ■

Arithmetic series

Now consider the *sum* of the first n terms of an arithmetic sequence. We denote this by S_n (called the nth partial sum). This sum can be written

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + [a_1 + (n - 1)d]$$

Another way to write this sum would be to add in reverse order, starting with the nth term, a_n, and subtracting multiples of the common difference d. Then we obtain

$$S_n = a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \cdots + [a_n - (n - 1)d]$$

Now if we add the corresponding terms of both members of the two equations, we obtain

$$\begin{array}{r} S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d] \\ + \underline{S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n - 1)d]} \\ 2S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)}_{n \text{ terms of } (a_1 + a_n)} \end{array}$$

We can write the right member as the product $n(a_1 + a_n)$ and the sum of these two equations is given by

$$2S_n = n(a_1 + a_n)$$

Dividing each member of the equation by 2, we obtain the sum of the first n terms.

Sum of the first n terms of an arithmetic sequence

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n - 1)d]$$

Note $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ is obtained by substituting $a_1 + (n - 1)d$ for a_n in $S_n = \frac{n}{2}(a_1 + a_n)$.

Example 12–3 D

1. Find the sum of the first twenty-five terms of the arithmetic sequence whose general term is $a_n = 3n + 5$.

   Given $a_n = 3n + 5$, we can determine $a_1 = 3(1) + 5 = 8$, $d = 3$ (the coefficient of n) and, using $a_n = a_1 + (n - 1)d$,

   $$\begin{aligned} a_{25} &= (8) + [(25) - 1](3) && \text{Replace } a_1 \text{ with 8, } n \text{ with 25, and } d \text{ with 3} \\ &= 8 + (24)(3) = 8 + 72 = 80 \end{aligned}$$

   Since $n = 25$, using the formula $S_n = \frac{n}{2}(a_1 + a_n)$,

$$S_{25} = \frac{(25)}{2}[(8) + (80)]$$ Replace n with 25, a_1 with 8, and a_{25} with 80

$$= \frac{25}{2}(88)$$ Perform indicated operations

$$= 25(44)$$
$$= 1{,}100$$

The sum of the first 25 terms is 1,100.

2. Find $\sum_{i=1}^{13} (3i - 1)$.

We want S_{13} (the thirteenth partial sum). Now $n = 13$, $a_1 = 3(1) - 1 = 2$, and $d = 3$ (the coefficient of i). We substitute into the formula $a_n = a_1 + (n - 1)d$ to determine a_{13}.

$$a_{13} = (2) + [(13) - 1](3)$$ Replace a_1 with 2, n with 13, and d with 3

$$= 2 + 12(3) = 2 + 36 = 38$$

Using $S_n = \frac{n}{2}(a_1 + a_n)$,

$$S_{13} = \frac{(13)}{2}[(2) + (38)]$$ Replace n with 13, a_1 with 2, and a_n with 38

$$= \frac{13}{2}(40)$$ Perform indicated operations

$$= 13(20)$$
$$= 260$$

Therefore $\sum_{i=1}^{13} (3i - 1) = 260$.

3. Find the sum of the first twelve terms of the arithmetic sequence whose first term $a_1 = -11$ and common difference $d = 4$.

Using the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$, we want S_{12}.

$$S_{12} = \frac{(12)}{2}[2(-11) + (12 - 1)(4)]$$ Replace n with 12, a_1 with -11, and d with 4

$$= 6\,[-22 + (11)4]$$ Perform indicated operations

$$= 6\,[-22 + 44]$$
$$= 6(22)$$
$$= 132$$

▶ ***Quick check*** Given the arithmetic sequence with $a_1 = 3$ and $a_n = -53$, find d and S_{29}.

Mastery points

Can you

- Determine if a sequence is an arithmetic sequence?
- Find the common difference of an arithmetic sequence?
- Find a specific term of an arithmetic sequence?
- Find the number of terms of a finite arithmetic sequence?
- Find the sum of a given number of terms in an arithmetic sequence?

Exercise 12–3

Determine whether or not the given sequence is arithmetic. If it is arithmetic, find the common difference d. See example 12–3 A.

Example $8,5,2,-1, \cdots$

Solution Since $5 - 8 = -3$, $2 - 5 = -3$, $-1 - 2 = -3$, the sequence is arithmetic and the common difference $d = -3$.

1. $2,3,4,5,6, \cdots$
2. $1,6,11,16,21, \cdots$
3. $-10,-8,-6,-4, \cdots$
4. $4,6,8,10,12, \cdots$
5. $3,5,8,10,12, \cdots$
6. $-16,-19,-22,-25, \cdots$
7. $\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2}, \cdots$
8. $-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2}, \cdots$
9. $-\frac{7}{3},-\frac{2}{3},1,\frac{8}{3}, \cdots$
10. $1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}, \cdots$

Find the indicated term of the arithmetic sequence having the following characteristics. See example 12–3 B.

Example Find the nineteenth term of the arithmetic sequence whose first term $a_1 = -2$ and the common difference $d = 3$.

Solution Using $a_n = a_1 + (n - 1)d$, we want a_{19} where $n = 19$.

$$\begin{aligned} a_{19} &= (-2) + [(19) - (1)](3) && \text{Replace } a_1 \text{ with } -2,\ n \text{ with } 19, \text{ and } d \text{ with } 3 \\ &= -2 + (18)3 && \text{Multiply and simplify} \\ &= -2 + 54 \\ &= 52 \end{aligned}$$

The nineteenth term of the sequence is 52.

11. $a_1 = 4$, $d = 5$; find a_{16}.
12. $a_1 = -4$, $d = 2$; find a_{21}.
13. $a_1 = -10$, $d = -3$; find a_{17}.
14. $a_1 = 6$, $d = -7$; find a_{23}.
15. $a_1 = 2$, $d = \frac{1}{3}$; find a_{12}.
16. $a_1 = 4$, $d = \frac{1}{2}$; find a_{10}.
17. $a_1 = \frac{1}{3}$, $d = \frac{2}{3}$; find a_{14}.
18. $a_1 = \frac{3}{4}$, $d = -\frac{1}{2}$; find a_{13}.
19. $5,9,13,17, \cdots$; find a_{16}.
20. $4,13,22,31, \cdots$; find a_{19}.
21. $-15,-10,-5,0, \cdots$; find a_{25}.
22. $-27,-25,-23,-21, \cdots$; find a_{20}.

Find the number of terms in the given finite arithmetic sequence. See example 12–3 C.

Example Find the number of terms of the arithmetic sequence $-5,-1,3,7, \cdots, 115$.

Solution We are given $a_1 = -5$ and $a_n = 115$, and we want n. Now $-1 - (-5) = 4$, so the common difference $d = 4$. Using $a_n = a_1 + (n - 1)d$,

$$\begin{aligned} (115) &= (-5) + (n - 1)(4) && \text{Replace } a_n \text{ with } 115,\ a_1 \text{ with } -5, \text{ and } d \text{ with } 4 \\ 115 &= 4n - 9 && \text{Simplify the right member} \\ 124 &= 4n && \text{Add 9 to each member} \\ 31 &= n && \text{Divide each member by 4} \end{aligned}$$

There are thirty-one terms in the arithmetic sequence.

23. $14,29,44,59, \cdots, 89$

24. $2,5,8, \cdots, 83$

25. $-3,-11,-19,-27, \cdots, -115$

26. $7,1,-5,-11, \cdots, -101$

27. $\frac{1}{2},0,-\frac{1}{2},-1, \cdots, -\frac{27}{2}$

$\boxed{28.}$ $\frac{5}{3},\frac{4}{3},1,\frac{2}{3}, \cdots, -2$

Find the indicated partial sum for each of the given arithmetic sequences. The last given term is the nth term, a_n. (That is, $a_{16} = 33$ in exercise 29.) See example 12–3 D–1.

Example Given $a_1 = 3$ and $a_n = -53$, find d and S_{29}.

Solution a. Using $a_n = a_1 + (n-1)d$,

$$(-53) = (3) + [(29) - 1]d \quad \text{Replace } a_n \text{ with } -53, a_1 \text{ with } 3, \text{ and } n \text{ with } 29$$

$$-56 = 28d \quad \text{Subtract 3 from each member, } 29 - 1 = 28$$

$$-2 = d \quad \text{Divide each member by 28}$$

b. Using $S_n = \frac{n}{2}(a_1 + a_n)$,

$$S_{29} = \frac{(29)}{2}[(3) + (-53)] \quad \text{Replace } n \text{ with } 29, a_1 \text{ with } 3, \text{ and } a_{29} \text{ with } -53$$

$$= \frac{29}{2}(-50) \quad \text{Combine in right member}$$

$$= 29(-25) \quad \text{Reduce in right member}$$

$$= -725$$

The sum of the first twenty-nine terms is -725.

29. $3,5,7, \cdots, 33$; find S_{16}.

30. $0,3,6, \cdots, 63$; find S_{22}.

31. $7,4,1, \cdots, -32$; find S_{14}.

32. $1,-7,-15, \cdots, -111$; find S_{15}.

$\boxed{33.}$ $\frac{1}{2},1,\frac{3}{2}, \cdots, 9$; find S_{18}.

34. $\frac{1}{4},1,\frac{7}{4}, \cdots, \frac{37}{4}$; find S_{13}.

35. $a_n = 2n + 5$; find S_{14}.

36. $a_n = 6n - 3$; find S_{15}.

$\boxed{37.}$ $a_n = 5 - n$; find S_{19}.

38. $a_n = 7 - 3n$; find S_{17}.

Find the indicated sum. See example 12–3 D–2.

39. $\sum_{k=1}^{15} (2k - 5)$

40. $\sum_{j=1}^{13} (3j + 9)$

41. $\sum_{i=1}^{22} (3 - 2i)$

42. $\sum_{i=1}^{19} (4 - i)$

43. $\sum_{j=1}^{17} \frac{1}{3}j$

$\boxed{44.}$ $\sum_{k=1}^{14} \frac{1}{2}k$

45. $\sum_{j=1}^{10} \left(\frac{3}{5}j - 2\right)$

46. $\sum_{i=1}^{11} \left(\frac{2}{3}i + 4\right)$

Find the indicated partial sum of the terms in the given arithmetic sequence. See example 12–3 D–3.

47. $2,8,14,22, \cdots$; find S_{13}.

48. $1,8,15,22, \cdots$; find S_{12}.

49. $24,19,14, \cdots$; find S_{15}.

50. $-6,-4,-2, \cdots$; find S_{14}.

$\boxed{51.}$ $\frac{1}{6},-\frac{5}{6},-\frac{11}{6}, \cdots$; find S_{11}.

52. $-1,-4,-7, \cdots$; find S_{16}.

Solve the following word problems.

53. A display of cans has 21 cans in the bottom row, 19 cans in the row above, 17 cans in the next row, and so on. How many cans are there if the top row contains 1 can? (*Hint:* $a_1 = 21$ and $d = -2$)

$\boxed{54.}$ A stock boy in a grocery store stacks a number of boxes of cereal so that there are 30 boxes in the first row, 27 boxes in the second row, 24 boxes in the third row, and so on. How many boxes of cereal does he have if there are 3 boxes in the top row?

55. In exercise 53, if there are 8 rows of cans, how many cans are in the display?

56. In exercise 54, if there are 9 rows of cereal, how many boxes of cereal are there?

57. Find the sum of the even integers from 2 to 116.

58. How many times will a clock strike in 12 hours if it strikes only on the hour?

59. A parachutist in free fall falls vertically 16 feet during the first second, 48 feet during the second second, 80 feet during the third second, and so on. How far will she fall during the eighth second? How far will she fall during the first 10 seconds?

60. Find the sum of the odd integers from 1 to 101.

61. Ron Line is offered a job as a mechanic starting at \$700 per month. If he is guaranteed a pay increase of \$10 per month every 3 months, what will his salary be after 8 years?

62. Neglecting air resistance, how long would it take before the parachutist pulls the rip cord to break her fall after falling 3,600 feet in exercise 59? (*Hint:* $S_n = 3{,}600$ and find n.)

63. In exercise 61, what total salary would Ron have earned in 8 years?

64. In exercise 61, how many years would it take for his salary to reach \$1,000 per month?

65. Kenny Kranz opened a savings account for his daughter by depositing \$50 on the day that she was born. On each subsequent birthday, he deposited \$30 more than the previous year. How much money was deposited on his daughter's eighteenth birthday?

66. In exercise 65, how much money (disregarding interest) had Kenny deposited for his daughter after her eighteenth birthday?

Review exercises

1. Subtract $\dfrac{3}{y-6} - \dfrac{2}{6-y}$. See section 4–3.

2. Given $f(x) = 5x - 3$, find $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$. See section 10–2.

3. Find the solution set of the logarithmic equation $\log_x 64 = 3$. See section 11–2.

4. Identify each equation as a parabola, circle, ellipse, or hyperbola. See section 9–4.

   a. $x^2 - y = 3x + 1$
   b. $3y^2 = 3x^2 + 1$
   c. $2x^2 + y^2 = 10$

5. Rationalize the denominator of $\dfrac{\sqrt{2}}{\sqrt{2} - \sqrt{3}}$. See section 5–5.

6. Reduce $\dfrac{3a^2 - 3}{3a^2 + 2a - 5}$ to lowest terms. See section 4–1.

12–4 ■ Geometric sequences and series

A geometric sequence

Suppose a man offers to rent you his house under the conditions that the rent will be figured daily as follows:

first day	1¢
second day	2¢
third day	4¢
fourth day	8¢
fifth day	16¢
sixth day	32¢
⋮	⋮

The daily rent on the house forms the sequence

$$1,2,4,8,16,32, \cdots$$

in which case each term, after the first, is obtained by multiplying the preceding term by the constant 2. Such a sequence is called a **geometric sequence**.

Definition of a geometric sequence

A **geometric sequence** is a sequence having the property that each term after the first term can be obtained by multiplying the preceding term by the same nonzero constant multiplier.

A geometric sequence is also called a **geometric progression.** We call the constant multiplier the **common ratio** since successive terms of the sequence form a "common ratio." We denote the common ratio by r and can obtain it by dividing any term after the first by the preceding term. This common ratio is the characteristic that distinguishes a geometric sequence from any other sequence.

Example 12–4 A

Determine whether the given sequence is geometric. If it is geometric, find the common ratio r.

1. $4,12,36,108, \cdots$
   Since $12 \div 4 = 3$, $36 \div 12 = 3$, and $108 \div 36 = 3$, the sequence is geometric and the common ratio $r = 3$.
2. $6,12,36,72, \cdots$
   Since $12 \div 6 = 2$ and $36 \div 12 = 3$, the sequence is not geometric.

▶ ***Quick check*** Determine whether the sequence $-1,2,-4,8, \cdots$ is geometric. If so, find the common ratio r. ■

General term of a geometric sequence

The definition of a geometric sequence shows that the first few terms of a general geometric sequence take the form

$$a_1, a_2 = a_1r, a_3 = a_1r^2, a_4 = a_1r^3, \cdots$$

from which we can determine the following about the general term of a geometric sequence.

General term of a geometric sequence

The general term of a geometric sequence with first term a_1 and common ratio r is given by

$$a_n = a_1r^{n-1}$$

Example 12–4 B

1. Given the sequence

   $2,10,50,250, \cdots$

   we can determine that this is a geometric sequence with the common ratio $r = 5$ because $\frac{10}{2} = 5$, $\frac{50}{10} = 5$, and $\frac{250}{50} = 5$. Since $a_1 = 2$, the general term of the geometric sequence is given by

   $a_n = 2(5)^{n-1}$

2. State the terms of the geometric sequence whose nth general term is

$$a_n = 3\left(\frac{1}{2}\right)^{n-1}$$

The terms of the sequence are

$$a_1 = 3\left(\frac{1}{2}\right)^0 = 3(1) = 3,\ a_2 = 3\left(\frac{1}{2}\right)^1 = \frac{3}{2},$$
$$a_3 = 3\left(\frac{1}{2}\right)^2 = 3\left(\frac{1}{4}\right) = \frac{3}{4},\ a_4 = 3\left(\frac{1}{2}\right)^3 = 3\left(\frac{1}{8}\right) = \frac{3}{8}, \text{ and so on.}$$

The sequence then is given by

$$3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \cdots, 3\left(\frac{1}{2}\right)^{n-1}, \cdots$$

where $a_1 = 3$ and $r = \frac{1}{2}$.

3. Find the sixth term, a_6, of the geometric sequence with $a_1 = 24$ and $r = -\frac{1}{2}$.

Using $a_n = a_1r^{n-1}$,

$$\begin{aligned} a_6 &= 24\left(-\frac{1}{2}\right)^{6-1} && \text{Replace } n \text{ with 6, } a_1 \text{ with 24, and } r \text{ with } -\tfrac{1}{2} \\ &= 24\left(-\frac{1}{2}\right)^5 \\ &= 24\left(-\frac{1}{32}\right) \\ &= -\frac{3}{4} \end{aligned}$$

The sixth term of the geometric sequence is $-\frac{3}{4}$.

▶ ***Quick check*** Find the sixth term of the geometric sequence with $a_1 = 27$ and $r = -\frac{1}{3}$. ■

Geometric series

Now consider the sum of the first n terms of a geometric sequence, denoted by S_n, and called the nth partial sum of the geometric sequence. With the geometric sequence

$$a_1, a_1r, a_1r^2, a_1r^3, \cdots, a_1r^{n-1}$$

is associated the geometric series

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} \tag{1}$$

If we multiply each member of this equation by the common ratio r, we obtain

$$rS_n = a_1r + a_1r^2 + a_1r^3 + a_1r^4 + \cdots + a_1r^{n-1} + a_1r^n \tag{2}$$

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Subtracting term by term equation (2) from equation (1), we obtain

$$\begin{array}{rl} S_n = & a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} \\ rS_n = & \qquad a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n \\ \hline S_n - rS_n = & a_1 - a_1r^n \end{array}$$

Thus

$$\begin{aligned} S_n - rS_n &= a_1 - a_1r^n \\ (1 - r)S_n &= a_1 - a_1r^n && \text{Factor } S_n \\ S_n &= \frac{a_1(1 - r^n)}{1 - r} && (r \neq 1) \end{aligned}$$

Note When $r = 1$, then

$$S_n = \underbrace{a_1 + a_1 + a_1 + \cdots + a_1}_{n \text{ terms}} = na_1$$

Sum of the first n terms of a geometric sequence

The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1)$$

where a_1 is the first term and r is the common ratio.

Example 12–4 C

1. Find the sum of the first seven terms of the geometric sequence whose first term $a_1 = 3$ and whose common ratio $r = 2$.

We want S_7 where $n = 7$, $a_1 = 3$, and $r = 2$. Using the formula

$$\begin{aligned} S_n &= \frac{a_1(1 - r^n)}{1 - r} \\ S_7 &= \frac{(3)[1 - (2^7)]}{1 - (2)} && \text{Replace } n \text{ with 7, } a_1 \text{ with 3, and } r \text{ with 2} \\ &= \frac{3(1 - 128)}{-1} \\ &= \frac{3(-127)}{-1} \\ &= \frac{-381}{-1} \\ &= 381 \end{aligned}$$

The sum of the first seven terms is 381.

2. Find $\sum_{k=1}^{4} 2(3)^k$.

The series is the fourth partial sum of a geometric sequence, where $a_1 = 2 \cdot 3 = 6$ and $r = 3$. We want S_4. Using the formula

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_4 = \frac{(6)[1 - (3^4)]}{1 - (3)} \quad \text{Replace } a_1 \text{ with 6, } r \text{ with 3, and } n \text{ with 4}$$

$$= \frac{6(1-81)}{-2} = \frac{6(-80)}{-2} = \frac{-480}{-2} = 240$$

Therefore $S_4 = \sum_{k=1}^{4} 2(3)^k = 240$.

▶ ***Quick check*** Find S_8 of the geometric sequence where $a_1 = 2$ and $r = 3$.

Mastery points

Can you

- Identify a geometric sequence?
- Find the common ratio of a geometric sequence?
- Find the general term of a geometric sequence?
- Write the terms of a geometric sequence?
- Find the indicated term of a given geometric sequence?
- Find the nth partial sum of a geometric sequence?

Exercise 12–4

Determine if the given terms form a geometric sequence. If they do, find the common ratio and write the next three terms of the sequence. See example 12–4 A.

Example $-1, 2, -4, 8, \cdots$

Solution Since $2 \div -1 = -2$, $-4 \div 2 = -2$, and $8 \div -4 = -2$, the sequence is geometric and the common ratio is $r = -2$. Successively multiplying by -2, the next 3 terms are $-16, 32, -64$.

1. $1, 3, 9, \cdots$
2. $6, 12, 24, \cdots$
3. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \cdots$
4. $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \cdots$
5. $4, -2, 1, \cdots$
6. $-1, \frac{1}{2}, -\frac{1}{4}, \cdots$
7. $6, -2, \frac{2}{3}, \cdots$
8. $12, 4, \frac{4}{3}, \cdots$
9. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$
10. $16, 48, 80, \cdots$

Find the general term, a_n, of the given geometric sequence. See example 12–4 B–1.

11. $3, 6, 12, \cdots$
12. $8, 12, 18, \cdots$
13. $27, -18, 12, \cdots$
14. $-81, 27, -9, \cdots$
15. $1, \sqrt{3}, 3, \cdots$
16. $32, 16\sqrt{2}, 16, \cdots$
17. $-\frac{1}{15}, \frac{1}{5}, -\frac{3}{5}, \cdots$

Find the indicated term of the geometric sequence having the following characteristics. See example 12–4 B–3.

Example Find the sixth term of the geometric sequence with $a_1 = 27$ and $r = -\frac{1}{3}$.

Solution Using $a_n = a_1r^{n-1}$

$a_6 = 27\left(-\frac{1}{3}\right)^{6-1}$ Replace n with 6, a_1 with 27, and r with $-\frac{1}{3}$

$= 27\left(-\frac{1}{3}\right)^5$

$= 27\left(-\frac{1}{243}\right)$ $\left(-\frac{1}{3}\right)^5 = -\frac{1}{243}$

$= -\frac{27}{243}$

$= -\frac{1}{9}$ Reduce to lowest terms

The sixth term of the geometric sequence is $-\frac{1}{9}$.

18. $a_1 = 3, r = 2$; find a_6.
19. $a_1 = 2, r = 3$; find a_5.
20. $a_1 = 16, r = \frac{1}{2}$; find a_7.
21. $a_1 = 81, r = \frac{1}{3}$; find a_4.
22. $a_1 = 1, r = -4$; find a_5.
23. $a_1 = 5, r = -2$; find a_6.
24. $a_1 = 25, r = -\frac{1}{5}$; find a_4.
$\boxed{25.}$ $a_1 = -32, r = -\frac{1}{4}$; find a_5.
$\boxed{26.}$ 3,18,108, $\cdots$; find a_6.
27. 9,18,36, $\cdots$; find a_7.
28. 10, −20, 40, $\cdots$; find a_8.
29. 7, −14, 28, $\cdots$; find a_9.

Find the indicated partial sum of the geometric sequence having the following characteristics. See example 12–4 C.

Example Find S_8 of the geometric sequence where $a_1 = 2$ and $r = 3$.

Solution Using $S_n = \frac{a_1(1 - r^n)}{1 - r}$

$S_8 = \frac{(2)[1 - (3^8)]}{1 - (3)}$ Replace n with 8, a_1 with 2, and r with 3

$= \frac{2(1 - 6{,}561)}{-2}$

$= -(-6{,}560)$

$= 6{,}560$

The sum of the first eight terms is 6,560.

30. $a_1 = 8, r = 2$; sum of the first six terms
31. $a_1 = 14, r = 3$; sum of the first five terms
32. $a_1 = -64, r = \frac{1}{4}$; sum of the first four terms
$\boxed{33.}$ $a_1 = -10, r = \frac{1}{5}$; sum of the first four terms
$\boxed{34.}$ 9,18,36, $\cdots$; find S_7.
35. 3,18,108, $\cdots$; find S_6.
36. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$; find S_5.

37. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$; find S_9.

38. $\frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \cdots$; find S_7.

39. $-5, 15, -45, \cdots$; find S_5.

See example 12–4 C–2.

40. $\sum_{i=1}^{9} 3^i$

41. $\sum_{j=1}^{8} 4^j$

$\boxed{42.}$ $\sum_{k=1}^{7} (-2)^k$

43. $\sum_{i=1}^{5} (-3)^i$

44. $\sum_{j=1}^{6} \left(\frac{1}{4}\right)^j$

45. $\sum_{k=1}^{7} \left(\frac{2}{3}\right)^k$

46. $\sum_{i=1}^{8} \left(-\frac{1}{3}\right)^i$

$\boxed{47.}$ $\sum_{k=1}^{6} 3\left(\frac{2}{5}\right)^k$

48. $\sum_{j=1}^{8} 4\left(\frac{2}{3}\right)^j$

49. $\sum_{i=1}^{6} -5\left(\frac{3}{5}\right)^i$

50. $\sum_{j=1}^{6} -3\left(-\frac{1}{3}\right)^j$

Solve the following word problems.

$\boxed{51.}$ A ball is dropped from a height of 9 feet. If on each rebound it rises two-thirds of the height from which it fell, what distance has it traveled when it strikes the ground for the sixth time?

52. A basketball rebounds to a height that is three-fourths of the height from which it fell. If the basketball is dropped initially from a height of 4 meters, what distance has it traveled when it strikes the floor for the fifth time?

53. On a visit to Las Vegas, Emmett Broughton doubled his bet each time that he lost. If his first bet was \$2 and he lost 8 consecutive bets, how much did he bet on the ninth bet?

54. In exercise 53, if Emmett loses his ninth bet also, how much will he have lost after his ninth loss?

55. In the first paragraph of this section, a man is paid monthly rent for his house at the rate of 1¢ the first day, 2¢ the second day, 4¢ the third day, and so on. At this rate, what would the rent for the house be for a month of 30 days?

56. A certain bacteria culture under a given condition triples in number each hour. If there were originally 1,000 bacteria, how many hours would it take the number of bacteria present in the culture to surpass 1 million?

57. A pump used to expel air from a tank removes one-fifth of what remains in the tank with each stroke. What part of the air in the tank has been removed after the fifth stroke?

58. An automobile depreciates in value each year by one-fifth of its value at the beginning of the year. If the automobile is purchased for \$8,000, what is its value at the end of the fourth year?

Review exercises

Perform the indicated operations. See section 3–2.

1. $(4x + y)^2$

2. $(3x - 2y)^3$

3. Complete the square of the expression $x^2 - 12x$. See section 6–2.

4. Evaluate the expression $\frac{p}{1-q}$ when $p = \frac{2}{3}$ and $q = \frac{1}{2}$. See section 1–5.

5. State the expression $\frac{3}{2-i}$ in the form $a + bi$. See section 5–7.

6. Find the solution set of the quadratic equation $2y^2 - y + 4 = 0$. See section 6–3.

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12–5 ■ Infinite geometric series

Consider the formula for the sum of the first n terms of a geometric sequence given by

$$S_n = \frac{a_1 - a_1r^n}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}$$

which can be written in the form

$$S_n = \frac{a_1}{1 - r}(1 - r^n)$$

Now let n get greater and greater and let $|r| < 1$. Recall that if $|r| < 1$, then $-1 < r < 1$. We can show that as n becomes increasingly large, r^n becomes closer to zero. To illustrate, if $r = \frac{1}{3}$, then

$$r^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}, \quad r^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}, \quad r^4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}, \text{ and so on}$$

As n increases, we can see $r^n = \left(\frac{1}{3}\right)^n$ approaches (gets closer and closer to) the value zero.

Now since $|r| < 1$ and r^n approaches the value zero as n increases, then

$$\frac{a_1}{1 - r}(1 - r^n)$$

approaches the value

$$\frac{a_1}{1 - r}(1 - 0) = \frac{a_1}{1 - r}$$

Definition of the sum of an infinite geometric series

If $|r| < 1$, the sum of the terms of an infinite geometric series, denoted by S_∞, is given by

$$S_\infty = \frac{a_1}{1 - r}$$

If $|r| \geq 1$, the sum does not exist, $a_1 \neq 0$.

In sigma notation, we write the preceding definition by

$$S_\infty = \sum_{i=1}^{\infty} a_1r^{i-1} = \frac{a_1}{1 - r}, \ |r| < 1$$

where the upper limit n is replaced by ∞ in the statement $S_n = \sum_{i=1}^{n} a_1 r^{i-1}$.

Example 12–5 A

1. Find the sum of the terms of an infinite geometric series such that $a_1 = 2$ and $r = \frac{1}{2}$.

We want S_∞ and use the formula

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{(2)}{1-\left(\frac{1}{2}\right)} && \text{Replace } a_1 \text{ with 2 and } r \text{ with } \tfrac{1}{2} \\ &= \frac{2}{\frac{1}{2}} \\ &= 4 \end{aligned}$$

The sum of the terms in the infinite geometric series is 4.

2. Find $\sum_{i=1}^{\infty} 2\left(\frac{1}{4}\right)^i$.

We want $S_\infty = \sum_{i=1}^{\infty} 2\left(\frac{1}{4}\right)^i$, in which $a_1 = 2\left(\frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}$ and $r = \frac{1}{4}$.

Using $S_\infty = \frac{a_1}{1-r}$

$$\begin{aligned} S_\infty &= \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{4}\right)} && \text{Replace } a_1 \text{ with } \tfrac{1}{2} \text{ and } r \text{ with } \tfrac{1}{4} \\ &= \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2} \cdot \frac{4}{3} \\ &= \frac{2}{3} \end{aligned}$$

Thus, $\sum_{i=1}^{\infty} 2\left(\frac{1}{4}\right)^i = \frac{2}{3}$.

▶ ***Quick check*** Find $\sum_{j=1}^{\infty} 12\left(\frac{1}{4}\right)^j$. ■

The sum of the terms of an infinite geometric series, $|r| < 1$, has some practical uses. We now consider two of the most common applications.

Example 12–5 B

1. Write the repeating decimal $0.2727\overline{27}$ as a rational number.

Now $0.2727\overline{27} = 0.27 + 0.0027 + 0.000027 + \cdots$. We have an infinite geometric series with $a_1 = 0.27$ and

$$r = \frac{0.0027}{0.27} = 0.01$$

Using the formula $S_\infty = \dfrac{a_1}{1 - r}$,

$$S_\infty = \frac{(0.27)}{1 - (0.01)} \quad \text{Replace } a_1 \text{ with } 0.27 \text{ and } r \text{ with } 0.01$$
$$= \frac{0.27}{0.99}$$
$$= \frac{27}{99}$$
$$= \frac{3}{11}$$

Thus the repeating decimal $0.272\overline{727} = \dfrac{3}{11}$.

2. A ball is dropped from a height of 12 meters. If each time it strikes the floor the ball rebounds to a height that is three-fourths of the height from which it fell, find the total distance that the ball travels before it comes to rest on the floor.

Let d be the total distance the ball travels. After the initial 12-meter drop, the ball will travel the same distance up and back down after each striking on the floor. Therefore

$a_1 = 12\left(\dfrac{3}{4}\right) = 9$ and $r = \dfrac{3}{4}$ and using the formula $S_\infty = \dfrac{a_1}{1 - r}$

$$S_\infty = \frac{(9)}{1 - \left(\frac{3}{4}\right)} \quad \text{Replace } a_1 \text{ with } 9 \text{ and } r \text{ with } \frac{3}{4}$$
$$= \frac{9}{\frac{1}{4}} = 9 \cdot \frac{4}{1}$$
$$= 36$$

Then $d = \underbrace{12}_{\text{Initial drop}} + \underbrace{2(36)}_{\text{Distance traveled up and down}} = 12 + 72 = 84.$

The ball would travel a distance of 84 meters before coming to rest.

▶ ***Quick check*** Write $0.363\overline{636}$ as a rational number in lowest terms. ■

Mastery points

Can you

- Find the sum of the terms of an infinite geometric series with $|r| < 1$?
- Express a repeating decimal as a rational number using $S_\infty = \dfrac{a_1}{1 - r}$, $|r| < 1$?

Exercise 12–5

Find the sum of the terms of the given infinite geometric series. If the series has no sum, indicate that condition. See example 12–5 A–1.

Example Find $\sum_{j=1}^{\infty} 12\left(\frac{1}{4}\right)^j$

Solution Using $S_\infty = \frac{a_1}{1-r}$ where $a_1 = 12\left(\frac{1}{4}\right) = 3$ and $r = \frac{1}{4}$,

$$S_\infty = \frac{(3)}{1 - \left(\frac{1}{4}\right)} \qquad \text{Replace } a_1 \text{ with 3 and } r \text{ with } \frac{1}{4}$$

$$= \frac{3}{\frac{3}{4}}$$

$$= 3 \cdot \frac{4}{3}$$

$$= 4$$

The sum of the terms in the series is 4.

1. $a_1 = 1, r = \frac{2}{3}$
2. $a_1 = 2, r = \frac{1}{3}$
3. $\boxed{3.}$ $a_1 = -3, r = \frac{1}{2}$
4. $a_1 = \frac{1}{5}, r = \frac{1}{10}$
5. $a_1 = \frac{3}{5}, r = \frac{1}{3}$
6. $\boxed{6.}$ $a_1 = 4, r = -\frac{1}{2}$
7. $a_1 = -\frac{5}{6}, r = -\frac{2}{3}$
8. $14 + 7 + \frac{7}{2} + \cdots$
9. $\boxed{9.}$ $12 + 4 + \frac{4}{3} + \cdots$
10. $3 + \frac{3}{4} + \frac{3}{16} + \cdots$
11. $4 + \frac{4}{5} + \frac{4}{25} + \cdots$
12. $1 + \frac{2}{3} + \frac{4}{9} + \cdots$
13. $\boxed{13.}$ $6 - 8 + \frac{32}{3} - \cdots$
14. $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1}$
15. $\sum_{j=1}^{\infty} \left(\frac{4}{5}\right)^j$
16. $\boxed{16.}$ $\sum_{k=1}^{\infty} \left(\frac{7}{8}\right)^{k+1}$
17. $\sum_{i=1}^{\infty} \left(-\frac{2}{3}\right)^i$
18. $\sum_{k=1}^{\infty} \left(-\frac{5}{3}\right)^k$
19. $\sum_{i=1}^{\infty} 3\left(\frac{1}{5}\right)^i$
20. $\sum_{j=1}^{\infty} 4\left(\frac{1}{6}\right)^{j-1}$

Express the given repeating decimal as a rational number in lowest terms. See example 12–5 B–1.

Example $0.363\overline{636}$

Solution We can write $0.3636\overline{36} = 0.36 + 0.0036 + 0.000036 + \cdots$

Then $a_1 = 0.36$ and $r = \dfrac{0.0036}{0.36} = 0.01$.

$$\begin{aligned}
\text{Using } S_\infty &= \frac{a_1}{1 - r} \\
&= \frac{(0.36)}{1 - (0.01)} && \text{Replace } a_1 \text{ with } 0.36 \text{ and } r \text{ with } 0.01 \\
&= \frac{0.36}{0.99} && \text{Subtract in denominator} \\
&= \frac{36}{99} && \text{Multiply numerator and denominator by } 100 \\
&= \frac{4}{11} && \text{Reduce to lowest terms}
\end{aligned}$$

The rational number equivalent of $0.3636\overline{36}$ is $\dfrac{4}{11}$.

21. $0.333\overline{3}$ **22.** $0.4242\overline{42}$ **$\boxed{23.}$** $0.2818\overline{18}\overline{1}$ **24.** $0.4727\overline{27}\overline{2}$ **25.** $0.03636\overline{36}$

Solve the following word problems. See example 12–5 B–2.

26. A ball returns to two-thirds of its previous height with each bounce. If the ball is dropped from a height of 6 feet, what is the total distance the ball will travel before coming to rest?

27. When a weight on an attached spring is dropped, it falls a distance of 30 inches before the spring stretches to its limit and the weight springs back up. If the weight rebounds to nine-tenths of the preceding distance it fell, through what total distance does the weight travel before coming to rest?

$\boxed{28.}$ A bob in a pendulum travels an arc length that is seven-eighths of its preceding arc length. If the first arc length is 16 centimeters, how far will the bob move before coming to rest?

29. If the first swing of a pendulum bob is 14 inches and each succeeding swing is five-sixths as long as the preceding one, what is the total distance the bob will travel before coming to rest?

30. A grant from an alumnus of Henry Ford Community College was such that the college was to receive \$30,000 the first year and two-thirds of the preceding year's donation each year thereafter. What was the total amount of money the college would receive from the alumnus?

31. Muriel Lakey's cat, Epu, receives 5 milligrams of a medicine at 2 P.M. If Epu is to receive four-fifths of the preceding dose of the medicine every hour thereafter, how many milligrams of medicine does Epu receive altogether?

Review exercises

1. Expand $(x + y)^3$ by multiplying $(x + y)(x + y)(x + y)$. See section 3–2.

Perform the indicated operations. See section 5–1.

2. $(a^{-4})^{1/2}$

3. $(a^6b^4)^{1/2}$

4. Sketch the graph of $f(x) = \sqrt{x + 1}$. See section 10–3.

5. Given $f(x) = x^2 + 2x + 1$ and $g(x) = 2x - 1$, find $f[g(x)]$. See section 10–2.

Completely factor the following. See section 3–7.

6. $a^4 - b^4$ **7.** $3x^2 - 27y^2$ **8.** $3x^3 + 24y^3$

12–6 ■ The binomial expansion

Consider the indicated product $(x + y)^n$, where n is a positive integer. By performing the indicated multiplication, we can obtain polynomial expressions for the positive integral powers of the binomial expression $x + y$. That is, we can multiply to show that

$$\begin{aligned}
(x + y)^1 &= x + y \\
(x + y)^2 &= x^2 + 2xy + y^2 \\
(x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x + y)^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{aligned}$$

and so on. Each of the polynomials thus obtained is called the **binomial expansion** of the related power of the binomial $x + y$. In this section, we shall develop a formula that will enable us to express any positive integral power of a binomial as a polynomial.

Before doing this, let us investigate the given expansions to determine the properties that will hold for the expansion of the general binomial $(x + y)^n$.

1. The first term of each expansion is x raised to the power of the binomial itself, x^n.
2. The second term of each expansion is of the form $nx^{n-1}y$.
3. As we proceed term by term from this point, the exponent of x decreases by 1 and the exponent of y increases by 1 with each succeeding term.
4. The next to last term is of the form nxy^{n-1}.
5. The last term of each expansion is y raised to the power of the binomial, y^n.
6. In each term, the sum of the exponents of x and y is always n.
7. There are $n + 1$ terms in each expansion.

Pascal's triangle

If we consider the coefficients of the five expansions stated previously, we can write them in a triangular pattern. (See figure 12–1.)

$$\begin{array}{lccccccccccc}
(x+y)^1 & & & & & 1 & & 1 & & & & \\
(x+y)^2 & & & & 1 & & 2 & & 1 & & & \\
(x+y)^3 & & & 1 & & 3 & & 3 & & 1 & & \\
(x+y)^4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
(x+y)^5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Figure 12–1

This pattern was used by a seventeenth-century French mathematician named Blaise Pascal (1623–62) and is called **Pascal's Triangle.** When the coefficients are thus arranged, it is possible to determine the coefficients of the next expansion.

Inspection reveals the following characteristics of Pascal's triangle:

1. The coefficients of the first and the last terms are always 1.
2. Each of the other coefficients is obtained by *adding the two numbers above it,* one to the left and one to the right.

To determine the coefficients of $(x + y)^6$, we use the coefficients of $(x + y)^5$ in our triangle to obtain the coefficients shown in figure 12–2.

$$\begin{array}{ccccccccccccc} & 1 & + & 5 & + & 10 & + & 10 & + & 5 & + & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

Figure 12–2

Therefore

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Factorial notation

Although it is possible to use Pascal's Triangle to determine the coefficients in any expansion $(x + y)^n$, where n is a positive integer, we need a more efficient way to do this for greater powers of the binomial. To do this, we must use **factorial notation.**

To write a product of n consecutive positive integers (starting with 1), we use the shorthand notation "$n!$," which is read "***n* factorial**" or "**factorial *n*.**"

Definiton of n factorial

$$n! = n(n - 1)(n - 2)(n - 3) \cdots (3)(2)(1)$$

Equivalently,

$$n! = (1)(2)(3) \cdots (n - 2)(n - 1)n$$

To illustrate,

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \quad \text{or} \quad 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

Note We agree that $0! = 1$. Then $0!$ and $1!$ both equal 1.

We now state the general binomial expansion of $(x + y)^n$ for any positive integer n.

$$(x + y)^n = x^n + \frac{n}{1!}x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^{n-4}y^4 + \cdots + \frac{n}{1!}xy^{n-1} + y^n$$

This statement is called the **binomial expansion** (or **binomial theorem**).

Example 12–6 A

1. Expand and simplify $(a + 3b)^4$.
   Applying the binomial expansion, we replace n by 4, x by a, and y by $3b$ to obtain the statement

$$\begin{aligned}(a + 3b)^4 &= a^4 + \frac{4}{1!}a^3(3b) + \frac{4\cdot 3}{2!}a^2(3b)^2 + \frac{4\cdot 3\cdot 2}{3!}a(3b)^3 + (3b)^4\\ &= a^4 + 4a^3(3b) + 6a^2(9b^2) + 4a(27b^3) + 81b^4\\ &= a^4 + 12a^3b + 54a^2b^2 + 108ab^3 + 81b^4\end{aligned}$$

2. Expand and simplify $(3c - 2d^2)^5 = [3c + (-2d^2)]^5$.
   From our binomial expansion $n = 5$, $x = 3c$, and $y = -2d^2$, so we substitute these values to obtain the statement

$$\begin{aligned}(3c - 2d^2)^5 &= (3c)^5 + \frac{5}{1!}(3c)^4(-2d^2) + \frac{5\cdot 4}{2!}(3c)^3(-2d^2)^2\\ &\quad + \frac{5\cdot 4\cdot 3}{3!}(3c)^2(-2d^2)^3\\ &\quad + \frac{5\cdot 4\cdot 3\cdot 2}{4!}(3c)(-2d^2)^4 + (-2d^2)^5\\ &= 243c^5 + 5(81c^4)(-2d^2) + 10(27c^3)(4d^4)\\ &\quad + 10(9c^2)(-8d^6) + 5(3c)(16d^8) + (-32d^{10})\\ &= 243c^5 - 810c^4d^2 + 1{,}080c^3d^4 - 720c^2d^6 + 240cd^8 - 32d^{10}\end{aligned}$$

▶ ***Quick check*** Expand and simplify $(a - 5b)^4$. ■

Finding the rth term of an expansion

Sometimes we wish to find one of the terms in the expansion and we would like to do this without expanding the binomial fully. We determine the rth term of the expansion using the following expression:

rth term of the binomial expansion $(x + y)^n$ is

$$\frac{n!}{[n - (r - 1)]!(r - 1)!}x^{n-(r-1)}y^{r-1}$$

In general, in the expansion of $(x + y)^n$, the term containing the variables $x^{n-k}y^k$ has coefficient $\dfrac{n!}{(n - k)!k!}$, which is sometimes written in the form $\dbinom{n}{k}$. That is, for the rth term in the expansion

$$\binom{n}{k} = \frac{n!}{(n - k)!k!}$$

where $k = r - 1$. To illustrate, by definition,

$$\begin{aligned}\binom{9}{5} &= \frac{9!}{(9 - 5)!5!} = \frac{9!}{4!\,5!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot 5!}{4\cdot 3\cdot 2\cdot 1\cdot 5!}\\ &= \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 9\cdot 2\cdot 7 = 126\end{aligned}$$

Note An alternative statement of the binomial expansion is $(x + y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$

Example 12–6 B

Find the sixth term in the expansion of $(2a + b)^9$.

Here $n = 9$, $r = 6$, $x = 2a$, and $y = b$. Since $r = 6$, then $r - 1 = 5$. We substitute 9 for n, 6 for r, 5 for $r - 1$, $2a$ for x, and b for y to obtain the expression for the sixth term as

$$\begin{aligned}\frac{9!}{(9-5)!5!}(2a)^{9-5}(b)^5 &= \frac{9!}{4!5!}(2a)^4(b)^5 \\ &= 126(16a^4)(b^5) = 2{,}016a^4b^5\end{aligned}$$

▶ ***Quick check*** Find the seventh term in the expansion of $(4a - 3b)^{10}$. ■

We can use the binomial expansion to approximate a power of a decimal number.

Example 12–6 C

Use the binomial expansion to evaluate $(2.01)^6$ correct to four decimal places. Expand to the first four terms.

Now $(2.01)^6 = (2 + 0.01)^6$ and applying the binomial expansion to this expression and expanding the first four terms,

$$\begin{aligned}(2 + 0.01)^6 &= 2^6 + \frac{6}{1!}(2)^5(0.01) + \frac{6 \cdot 5}{2!}(2)^4(0.01)^2 + \frac{6 \cdot 5 \cdot 4}{3!}(2)^3(0.01)^3 \\ &= 64 + 6(32)(0.01) + 15(16)(0.0001) + 20(8)(0.000001) \\ &= 64 + 192(0.01) + 240(0.0001) + 160(0.000001) \\ &= 64 + 1.92 + 0.024 + 0.00016 = 65.94416\end{aligned}$$

Rounding to four decimal places, $(2.01)^6 \approx 65.9442$. ■

Mastery points

Can you

- Evaluate factorial expressions?
- Expand and simplify a binomial expression raised to any positive integer power n?
- Determine a specific term in the expansion of a binomial?
- Use the binomial expansion to approximate the value of a decimal number to some positive integer power n?

Exercise 12–6

Expand and simplify each expression.

1. $6!$ **2.** $7!$ **3.** $\frac{12!}{8!}$ **4.** $\frac{9!}{7!}$ $\boxed{5.}$ $\frac{10!}{9!}$

6. $\frac{15!}{14!}$ **7.** $\frac{8!}{2!6!}$ **8.** $\frac{7!}{4!3!}$ **9.** $\frac{10!}{3!7!}$ **10.** $\frac{13!}{5!8!}$

Expand and simplify the following binomials using Pascal's Triangle. See example 12–6 A.

Example $(a - 5b)^4$

Solution In our expansion of $(x + y)^4$, let $x = a$ and $y = -5b$ and use the coefficients 1, 4, 6, 4, 1.

$$\begin{aligned}(a - 5b)^4 &= 1(a)^4 + 4(a)^3(-5b) + 6(a)^2(-5b)^2 + 4(a)(-5b)^3 + 1(-5b)^4 \\ &= a^4 - 20a^3b + 150a^2b^2 - 500ab^3 + 625b^4\end{aligned}$$

$\boxed{11.}$ $(a - 3)^4$ **12.** $(b + 2)^5$ **13.** $(p + q)^6$ **14.** $(a - b)^7$ **15.** $(2a + 3)^4$

16. $(3b + 2)^5$ **17.** $\left(\frac{p}{2} - q\right)^6$ **18.** $\left(2r - \frac{q}{3}\right)^4$ $\boxed{19.}$ $(a^2 + b^2)^5$

Find the indicated term of each binomial expansion. See example 12–6 B.

Example Find the seventh term in the expansion of $(4a - 3b)^{10}$.

Solution Here $n = 10$, $r = 7$, $x = 4a$, and $y = -3b$. Since $r = 7$, then $r - 1 = 6$. Using

$$\frac{n!}{[n - (r - 1)]!(r - 1)!}x^{n - (r - 1)}y^{r - 1}$$

$$\begin{aligned}\frac{10!}{(10 - 6)!6!}(4a)^{10 - 6}(-3b)^6 &= \frac{10!}{4!6!}(4a)^4(-3b)^6 \\ &= 210(256a^4)(729b^6) \\ &= 39{,}191{,}040a^4b^6\end{aligned}$$

20. $(a + b)^{13}$, seventh term **21.** $(a - b)^{14}$, sixth term **22.** $(p + 3)^{11}$, eighth term

23. $(q - 2)^{12}$, fifth term **24.** $(r - 2s)^{10}$, fifth term $\boxed{25.}$ $(6 - k)^9$, seventh term

Use the binomial expansion to calculate the following expressions correct to four decimal places. Expand to the first four terms. See example 12–6 C.

26. $(1.01)^8$ $\boxed{27.}$ $(1.002)^{13}$ **28.** $(2.02)^7$ **29.** $(0.97)^5$

30. In the expansion of $\left(p^2 - \frac{1}{4}\right)^{12}$, find the term involving p^{10}.

$\boxed{31.}$ Find the middle term in the expansion of $(a + \sqrt{a})^{12}$.

Evaluate the following. See example 12–6 B.

32. $\binom{5}{2}$ **33.** $\binom{8}{4}$ **34.** $\binom{12}{7}$

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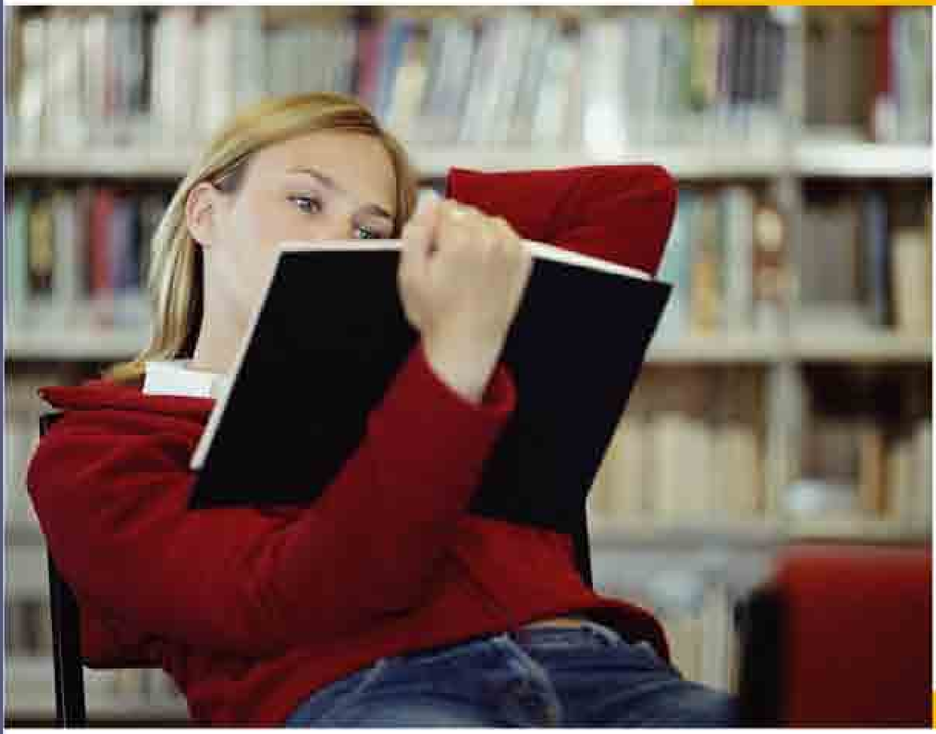

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Chapter 12 lead-in problem

A sky diver falls 10 meters during the first second, 20 meters during the second second, 30 meters during the third second, and so on. How many meters will the diver fall during the eleventh second?

Solution

This is an arithmetic sequence where we use the formula $a_n = a_1 + (n - 1)d$, where a_1 is the first term of the sequence, n is the number of the term we wish, a_{11}, and d is the common difference between each second. Thus,

$a_1 = 10$, $n = 11$, and $d = 10$.

$$a_{11} = (10) + [(11) - 1](10)$$

Replace a_1 with 10, n with 11, and d with 10

$$= 10 + 10(10)$$
$$= 10 + 100$$
$$= 110$$

The sky diver will fall 110 meters during the eleventh second.

Chapter 12 summary

1. An infinite **sequence** is a function whose domain is the set of the positive integers.
2. A sequence is **finite** when its domain is the set $\{1,2,3, \cdots ,n\}$ for some fixed n and **infinite** when the domain is the set of positive integers.
3. A **series** is the sum of the first r terms of a sequence.
4. The sum of the first n terms of a sequence whose general term is a_n, in **sigma notation**, is given by

   $$\sum_{i=1}^{n} a_i$$

   where i is the **index of summation,** 1 is the **lower limit,** and n is the **upper limit** of summation.
5. An **arithmetic sequence** is a sequence in which each term after the first differs from the preceding term by the same common difference d.
6. The general term a_n of an arithmetic sequence is given by

   $$a_n = a_1 + (n - 1)d$$

   where a_1 is the first term and d is the common difference.
7. The nth partial sum S_n of the terms of an arithmetic sequence is given by

   $$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n - 1)d]$$

8. A **geometric sequence** is a sequence in which each term after the first term can be obtained by multiplying the preceding term by the same nonzero constant multiplier, called the **common ratio** and denoted by r.
9. The general term of a geometric sequence with first term a_1 and common ratio r is given by

   $$a_n = a_1r^{n-1}$$

10. The nth partial sum S_n of the terms of a geometric sequence is given by

    $$S_n = \frac{a_1 - a_1r^n}{1 - r} = \frac{a_1(1 - r^n)}{1 - r} (r \neq 1)$$

11. The sum of the terms in an infinite geometric series is given by

    $$S_n = \frac{a_1}{1 - r}, |r| < 1$$

12. The product "n factorial," denoted by $n!$, is defined by

    $$n! = n(n - 1)(n - 2) \cdots (3)(2)(1)$$

13. The **binomial expansion** of the binomial $(x + y)^n$ is given by $(x + y)^n$

    $$= x^n + \frac{n}{1!}x^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots \frac{n}{1!}xy^{n-1} + y^n$$

14. $$\binom{n}{k} = \frac{n!}{(n - k)!k!}$$

Chapter 12 error analysis

1. Terms of a sequence
   Example: Given $a_n = (-1)(2n + 3)$, find a_1, a_2, and a_3.
   $a_1 = (-1)[2(1) + 3] = -5$
   $a_2 = (-1)[2(2) + 3] = 7$
   $a_3 = (-1)[2(3) + 3] = -9$
   Correct answer: $-5, -7, -9$
   What error was made? (*see page 514*)
2. Summation notation
   Example:
   $$\sum_{i=2}^{5}(2i + 5) = [2(1) + 5] + [2(2) + 5] + [2(3) + 5] + [2(4) + 5] + [2(5) + 5] = 7 + 9 + 11 + 13 + 15 = 55$$
   Correct answer: 48
   What error was made? (*see page 519*)
3. Summation notation
   Example: Write the partial sum $3 + 8 + 13 + 18$ in sigma notation $\sum_{i=1}^{4}(4i - 1)$
   Correct answer: $\sum_{i=1}^{4}(5i - 2)$
   What error was made? (*see page 520*)
4. Arithmetic sequence
   Example: The sequence $3, -1, -6, -12, -19, \ldots$ is arithmetic.
   Correct answer: The sequence is not arithmetic.
   What error was made? (*see page 523*)
5. Geometric sequence
   Example: The sequence $5, \frac{5}{2}, \frac{5}{6}, \frac{5}{24}, \frac{5}{120}, \ldots$ is geometric.
   Correct answer: The sequence is not geometric.
   What error was made? (*see page 530*)
6. Geometric series
   Example: Find $\sum_{i=1}^{3} 3(2)^i$
   $$\sum_{i=1}^{3} 3(2)^i = 3 + 12 + 24 = 39$$
   Correct answer: $\sum_{i=1}^{3} 3(2)^i = 42$
   What error was made? (*see page 532*)
7. Geometric series
   Example: Find $\sum_{k=1}^{5} 3(3)^k$
   Using $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$, where $n = 5$, $a_1 = 9$, and $r = 3$,
   $$S_5 = \frac{9(1 - 3^5)}{1 - 3} = \frac{9(2^5)}{2} = 9(2^4) = 144$$
   Correct answer: $\sum_{k=1}^{5} 3(3)^k = 1{,}089$
   What error was made? (*see page 532*)
8. Infinite geometric series
   Example: Find $\sum_{i=1}^{\infty} 3\left(\frac{1}{5}\right)^{i-1}$
   Using $S_\infty = \dfrac{a_1}{1 - r}$, where $a_1 = \dfrac{3}{5}$ and $r = \dfrac{1}{5}$,
   $$S_\infty = \frac{\frac{3}{5}}{1 - \frac{1}{5}} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}.$$
   Correct answer: $\sum_{i=1}^{\infty} 3\left(\frac{1}{5}\right)^{i-1} = \frac{15}{4}$
   What error was made? (*see page 537*)
9. Factorial notation
   Example: $\binom{7}{3} = \dfrac{7!}{3!} = \dfrac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!}$
   $= 7 \cdot 6 \cdot 5 \cdot 4 = 840$
   Correct answer: 35
   What error was made? (*see page 543*)
10. Sums of radical expressions
    Example: $\sqrt{16 + 36} = \sqrt{16} + \sqrt{36} = 4 + 6 = 10$
    Correct answer: $2\sqrt{13}$
    What error was made? (*see page 237*)

Chapter 12 critical thinking

Write an algebraic expression for the following relationship. Will this relationship always be true?

$$(1)^2 + (2)^2 = (3)^2 - (2)^2$$
$$(2)^2 + (3)^2 = (7)^2 - (6)^2$$
$$(3)^2 + (4)^2 = (13)^2 - (12)^2$$
$$(4)^2 + (5)^2 = (21)^2 - (20)^2$$

Chapter 12 review

[12–1]

Write the first five terms of each sequence whose general term a_n is given.

1. $a_n = 4n + 3$

2. $a_n = \dfrac{5n}{2n - 1}$

3. $a_n = (-1)^n \cdot \dfrac{4}{2n + 5}$

4. $a_n = (-1)^{n+1} \cdot 2^n$

Find the indicated term of the sequence whose general term a_n is given.

5. $a_n = 4 - 3n$, find a_6.

6. $a_n = (-1)^n(3n-4)$, find a_7.

7. $a_n = \dfrac{3^{n+1}}{2n}$, find a_9.

8. $a_n = (-1)^{n-1} \cdot \dfrac{2^n + 1}{3^n}$, find a_{11}.

Given the following sequences, find an expression for the general term a_n.

9. $5, 7, 9, 11, \cdots$

10. $3, 8, 13, 18, \cdots$

11. $\dfrac{2}{3}, \dfrac{3}{7}, \dfrac{4}{11}, \dfrac{5}{15}, \cdots$

12. $-4, 9, -14, 19, \cdots$

13. Dockage fees for a boat at a marina are \$3.00 for the first night, \$3.25 for the second night, \$3.50 for the third night, and so on. Write an expression for the general term of the sequence. How much did it cost to dock the boat on the eighth night?

[12–2]

Expand each indicated sum and find the sum.

14. $\sum_{i=1}^{4} (4i - 1)$

15. $\sum_{i=1}^{6} i(i + 5)$

16. $\sum_{k=1}^{5} \dfrac{k^2}{k + 1}$

17. $\sum_{j=1}^{6} (-1)^j \cdot \dfrac{4}{5j}$

Write each sum in sigma notation.

18. $5 + 8 + 11 + 14$

19. $\dfrac{4}{5} - \dfrac{5}{6} + \dfrac{6}{7} - \dfrac{7}{8} + \dfrac{8}{9}$

[12–3]

Find the indicated term of each arithmetic sequence having the following characteristics.

20. $a_1 = 5$, $d = 4$; find a_{15}.

21. $a_1 = -3$, $d = 5$; find a_{17}.

22. $3, 7, 11, 15, \cdots$; find a_{21}.

23. $-6, -8, -10, -12$; find a_{19}.

Find the number of terms in each given finite arithmetic sequence.

24. $-2, 3, 8, \cdots, 68$

25. $4, 0, -4, -8, \cdots, -76$

Find the indicated partial sum of each given arithmetic sequence.

26. $-3, 3, 9, \cdots, 81$; find S_{15}.

27. $10, 7, 4, \cdots, -50$; find S_{21}.

28. $\sum_{j=1}^{29} \dfrac{2}{3} j$

29. $\sum_{k=1}^{25} \left(\dfrac{1}{2} k + 1\right)$

30. Company B starts offers of a beginning wage of \$12,000 with a raise of \$450 each year thereafter. What would the wage be after 11 years?

[12–4]

Find the general term a_n of each given geometric sequence.

31. $3, 6, 12, \cdots$

32. $-\frac{3}{4}, \frac{9}{16}, -\frac{27}{64}, \cdots$

Find the indicated term of each geometric sequence having the following characteristics.

33. $a_1 = 5, r = 3$; find a_5.

34. $a_1 = -24, r = \frac{1}{3}$; find a_4.

35. $a_1 = 36, r = -\frac{2}{3}$; find a_3.

36. $-9, 18, -36, \cdots$; find a_7.

Find the indicated partial sum of each geometric sequence having the following characteristics.

37. $a_1 = 3, r = 3$; find S_5.

38. $a_1 = -24, r = \frac{1}{2}$; find S_6.

39. $\sum_{j=1}^{5} \left(-\frac{3}{4}\right)^j$

40. $\sum_{k=1}^{7} 4\left(\frac{1}{3}\right)^k$

[12–5]

Find the sum of the terms of each given infinite geometric series.

41. $a_1 = 3, r = \frac{3}{4}$

42. $a_1 = -2, r = \frac{1}{5}$

43. $\sum_{j=1}^{\infty} 3\left(-\frac{1}{4}\right)^j$

44. $\sum_{k=1}^{\infty} \left(-\frac{2}{3}\right)\left(-\frac{1}{5}\right)^{k+1}$

Use the infinite geometric series to write each repeating decimal as a rational number.

45. $0.3535\overline{35}$

46. $0.4323\overline{232}$

47. A boat at anchor experiences a series of waves, each wave having 25% less amplitude (height) than the previous one. If the first wave has amplitude 3 meters, how much vertical distance does the boat travel before coming to rest? (*Hint:* The boat travels *up* and *down* the same distance with each wave.)

[12–6]

Expand and simplify each binomial.

48. $(x + 5)^7$

49. $(2a - 3b)^5$

50. $\left(\frac{1}{2}a - 3b\right)^4$

Find the indicated term in the expansion of each given binomial.

51. $(a - 4)^{11}$, fifth term

52. $(3a + b)^{14}$, seventh term

53. $(2x + 3y)^{12}$, the term in which y^2 appears

54. The pendulum on a Seth Thomas antique regulator clock swings such that, after the first swing, each swing the pendulum travels is three-fourths of the previous swing. If it travels 2 feet on the first swing, how far does it travel on the fourth swing?

55. Evaluate $\binom{13}{8}$.

Final examination

[1–1] 1. Given $\{y|y$ is an integer between -8 and $4\}$, list the elements in the set.

[1–1] 2. Given $A = \{-1,2,4,7\}$, $B = \{0,1,2,7,9\}$, and $C = \{-4,-1,0,9\}$, find (a) $A \cap B$, (b) $A \cup C$, (c) $(B \cup C) \cap \emptyset$.

[1–1] 3. Given the set $\{x| -4 \leq x < 9\}$, list the integers in the set.

[1–4] 4. Perform the indicated operations and simplify. $5 - \{6 - [3 + 18 \div 2 - 3^2] - (4 + 7)\}$

[3–3] 5. Simplify the following expressions. Leave all answers with positive exponents.
a. $(-a^3b^{-3})(2a^4b^{-2})$
b. $(-2a^{-3}b^2)^{-2}$
c. $\dfrac{a^{-4}b^3}{a^2b^{-1}}$

[3–2] 6. Multiply as indicated.
a. $(2y + 7)^2$
b. $(4x + 2y)(4x - 2y)$
c. $(x - 2)(x^2 + 2x + 4)$

Completely factor the following expressions.

[3–4] 7. $12xy - 4x^2y^3 + 8x^3y^2$

[3–6] 8. $7y^2 - 34y - 5$

[3–6] 9. $4a^2 - 20ab + 25b^2$

[3–7] 10. $8x^2 - 50y^2$

[3–7] 11. $16a^3 - 2b^3$

[3–4] 12. $6ax - 3ay - 2bx + by$

Find the solution set of the following equations and inequalities.

[2–1] 13. $5(3y - 1) + 2y = 8(2 - y)$

[2–4] 14. $3(2x + 3) < 4(x - 5)$

[2–5] 15. $-9 < 4x + 1 \leq 5$

[2–4] 16. $|2x - 5| = 3$

[2–5] 17. $|5 - 3x| < 4$

[2–5] 18. $|5x - 4| \geq 6$

[6–1] 19. $x^2 - 5x = 24$

[4–7] 20. $\dfrac{3}{x - 1} - \dfrac{4}{3} = \dfrac{6}{x - 1}$

[4–3] 21. Add $\dfrac{5}{2a - 1} + \dfrac{6}{4a + 3}$.

[4–3] 22. Subtract $\dfrac{2y + 1}{y^2 - y - 42} - \dfrac{y + 1}{y^2 - 36}$.

[4–2] 23. Divide $\dfrac{3a^2 - 13a + 4}{a^2 + 2a + 1} \div \dfrac{a^2 - 8a + 16}{a^2 - 1}$.

[4–4] 24. Simplify the complex fraction $\dfrac{\frac{4}{a} - \frac{2}{b}}{\frac{2}{b} - \frac{1}{a}}$.

[4–6] 25. Divide $(3x^3 + 4x^2 - 2x + 1) \div (x - 3)$ using synthetic division.

[5–5] 26. Combine the expression $3\sqrt{75} + 2\sqrt{27} - \sqrt{48}$.

[5–7] 27. Rationalize the denominator $\dfrac{2 + i}{3 - 2i}$.

[5–6] 28. Multiply $(3 - 2\sqrt{5})(4 + 3\sqrt{2})$.

[6–3] 29. Find the solution set of the quadratic equation $3p^2 - 2 = 7p$.

[6–5] 30. Find the solution set of the radical equation $\sqrt{x - 1} = x - 3$. Indicate extraneous solutions.

[6–7] 31. Find the solution set of the quadratic inequality $y^2 \geq 4y - 3$.

[7–3] 32. Find the equation of the line having the following conditions. Write the equation in standard form.
a. Through points $(1,2)$ and $(-3,-4)$
b. Through $(4,-3)$ and perpendicular to the line $y - 4x = 3$
c. Through $(3,-2)$ and having slope $-\dfrac{2}{3}$

[7–3] 33. Find the slope and y-intercept of the line $5y - 3x = 9$.

[10–2] 34. Given $f(x) = 5x - 3$ and $g(x) = x^2 - x + 1$, find
a. $f(-3)$
b. $g(4)$
c. $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$

Sketch the graph of the following equations and inequalities.

[7–3] 35. $2y - 5x = 10$

[9–1] 36. $y = x^2 - 3x - 10$

[7–4] 37. $4y - 3x < -24$

[9–3] 38. $5x^2 + y^2 = 20$

[9–3] 39. Determine if the given equation represents a circle, a parabola, an ellipse, or a hyperbola.
a. $x^2 + y^2 - 2x + 4y - 3 = 0$
b. $2y^2 + 6 = x^2$
c. $4x - x^2 = y$
d. $8y^2 = 6 - 5x^2$

[8–1] 40. Find the solution set of the system of equations
$3y + 5x = 1$
$2y - 3x = -3$.

[8–3] 41. Evaluate the determinate $\begin{vmatrix} 4 & -1 & 3 \\ 5 & 0 & -2 \\ 3 & 6 & 1 \end{vmatrix}$

[8–5] 42. Find the solution set of the system of equations $\begin{matrix} x - 3y = 4 \\ 2x + 5y = 3 \end{matrix}$ by determinants using Cramer's Rule.

[11–3] 43. Write the expression $\log_b 5 + \log_b 6 - 3 \log_b 2$ as a logarithm of a single number.

[11–5] 44. Find $\log_3 7$ using the common logarithms.

[11–6] 45. Find the solution set of the equation $3^{2-x} = 4$. Round to the nearest tenth.

[11–5] 46. Find ℓn 36. Round off to four decimal places.

[12–1] 47. Write the sum $3 + 9 + 15 + 21 + 27$ in summation notation.

[12–2] 48. Find the indicated sum $\sum_{i=1}^{7} (2i - 1)$.

[12–3] 49. Given $a_1 = -2$ and $d = -3$, find a_{15} of the arithmetic sequence.

[12–3] 50. Find S_{26} of the arithmetic sequence $\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$.

[12–4] 51. Find a_4 of the geometric sequence given $a_1 = \frac{1}{2}$ and $r = -\frac{1}{3}$.

[12–5] 52. Find $\sum_{i=1}^{\infty} 2\left(-\frac{1}{2}\right)^i$.

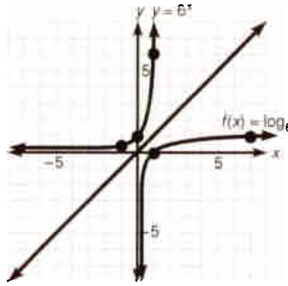
[12–6] 53. Expand the binomial $(3x - 2y)^4$.

[12–6] 54. Find the seventh term of the expansion of $(a - 5)^{12}$.

[12–5] 55. Find the rational equivalent of $0.234234\overline{234}$.

[12–6] 56. Evaluate $\binom{11}{5}$.

2. a. $\{4\}$ **b.** $\left\{\frac{7}{8}\right\}$ **3.**



4. $\log_4\frac{1}{64} = -3$ **5.** $\left(\frac{1}{3}\right)^{-3} = 27$ **6.** 3 **7.** -4 **8.** $\frac{1}{3}$

9. $\{5\}$ **10.** $\left\{\frac{1}{243}\right\}$ **11.** $\{4\sqrt{2}\}$ **12.** $x < -5$ or $x > 3$

13. $\log_b 7 + 3\log_b 2$ **14.** $\log_4 5 - \log_4 2 - \log_4 3$

15. $2\log_5 3 - 2\log_5 2$ **16.** $\log_5 28$ **17.** $\log_6 5$ **18.** $\log_b\frac{x^4y^2}{z}$

19. $\log_4\left(\frac{x+3}{x-4}\right)$ **20.** $\log_b\left(\frac{x^3(2x-1)}{(x+1)^3}\right)$ **21.** 1.3423; $b^{1.3423} = 22$ **22.** 2.0826; $b^{2.0826} = 121$ **23.** 1.9821; $b^{1.9821} = 96$

24. 0.3471; $b^{0.3491} = \sqrt[3]{11}$ **25.** -0.8652; $b^{-0.8652} = \frac{3}{22}$

26. 1.1710; $b^{1.1710} = \left(\frac{27}{\sqrt[4]{11}}\right)$ **27.** $\{48\}$ **28.** $\left\{\frac{2}{9}\right\}$ **29.** $\left\{\frac{23}{4}\right\}$

30. $\left\{\frac{83}{26}\right\}$ **31.** 2.5340 **32.** 5.7050 **33.** -2.1331

34. -0.7784 **35.** 15.2 yr **36.** 1.09 **37.** 0.45 **38.** 1.2 days **39.** 2.0 days **40.** $\{1.76\}$ **41.** $\{-0.79\}$ **42.** $\{2.81\}$ **43.** $\{0.40\}$ **44.** $\{-1.04\}$ **45.** 2.20 hours

Chapter 11 cumulative test

1. $\{-3,-2,-1,0,1,2,3,4,5\}$ **2.** -28 **3.** -81

4. $4a^2 - 4ab + b^2$ **5.** $25x^2 - 9y^2$ **6.** $12y^2 - 14y - 10$

7. $-18a^5b^3$ **8.** $-\frac{27b^6}{a^3c^9}$ **9.** $\frac{a^{12}}{9b^{10}}$ **10.** $\{1\}$ **11.** $\{y \mid y \le 2\}$

12. $\left\{x \mid -2 \le x < \frac{5}{2}\right\}$ **13.** $\left\{\frac{5}{4},\frac{1}{4}\right\}$ **14.** $\left\{x \mid -\frac{2}{3} \le x \le 2\right\}$

15. $\{x \mid x < -11 \text{ or } x > 1\}$ **16.** $\left\{\frac{3}{2},-1\right\}$ **17.** $\left\{\frac{5}{4}\right\}$

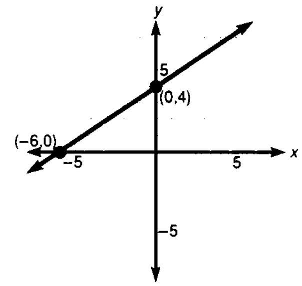
18. $\frac{5a+3b}{4a^2b^2}$ **19.** $\frac{15y}{y-7}$ **20.** $\frac{x^2-9x+20}{x^2+2x-15}$ **21.** 1

22. $13 - 4\sqrt{3}$ **23.** 25 **24.** $-i$ **25.** $7\sqrt{2}$ **26.** $\frac{2\sqrt{10}+4}{3}$

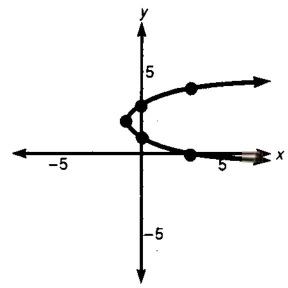
27. $\left\{\frac{3+\sqrt{37}}{2},\frac{3-\sqrt{37}}{2}\right\}$ **28.** $\{8\}$; -1 is extraneous

29. $\{z \mid -5 \le z \le 8\} = [-5,8]$ **30. a.** $2x - y = -6$ **b.** $2x - 3y = -42$ **c.** $4x - 3y = 29$ **31. a.** 26 **b.** -25 **c.** -44 **d.** 5

32.



33.



34. a. circle **b.** parabola **c.** hyperbola **d.** ellipse

35. $\left\{\left(\frac{7}{26},\frac{4}{13}\right)\right\}$ **36.** $\{2\}$ **37. a.** 2 **b.** $\frac{1}{36}$

38. 1.36 **39.** $\left\{\frac{17}{8}\right\}$ **40.** $\log_b\left(\frac{8}{125}\right)$

Chapter 12

Exercise 12-1

Answers to odd-numbered problems

1. 7,11,15,19,23 **3.** $\frac{2}{3},\frac{1}{3},\frac{2}{9},\frac{1}{6},\frac{2}{15}$ **5.** $-6,\frac{7}{2},\frac{8}{5},\frac{9}{8},\frac{10}{11}$

7. $\frac{2}{5},\frac{2}{5},\frac{8}{15},\frac{4}{5},\frac{32}{25}$ **9.** $-1,7,-13,19,-25$ **11.** $1,-\frac{9}{5},3,-\frac{81}{17},\frac{81}{11}$

13. 1,1,1,1,1 **15.** 32 **17.** $\frac{1}{111}$ **19.** $\frac{65}{41}$ **21.** 89

23. -247 **25.** $a_n = 2n + 4$ **27.** $a_n = 5n - 3$

29. $a_n = n^3$ **31.** $a_n = \frac{1}{2^{n-1}}$ **33.** $a_n = \frac{n+2}{2n+3}$

35. $a_n = (-1)^n(4n+2)$ **37. a.** 27,000 **b.** 243,000 **c.** $1{,}000(3^n)$ **39.** $\frac{5}{2}$ ft; $\frac{5}{32}$ ft; $5\left(\frac{1}{2}\right)^{n-1}$ **41. a.** \$16,000; \$17,500; \$19,000; \$20,500; \$22,000; \$23,500 **b.** \$16,000 + 1500 (n − 1) **c.** \$44,500

Solutions to trial exercise problems

11. $a_n = (-1)^{n-1} \cdot \frac{3^n}{2^n + 1}$; $a_1 = (-1)^{1-1} \cdot \frac{3}{2+1} = 1 \cdot \frac{3}{3} = 1$;

$a_2 = (-1)^{2-1} \cdot \frac{3^2}{2^2+1} = (-1) \cdot \frac{9}{4+1} = -\frac{9}{5}$;

$a_3 = (-1)^{3-1} \cdot \frac{3^3}{2^3+1} = 1 \cdot \frac{27}{8+1} = 3$;

$a_4 = (-1)^{4-1} \cdot \frac{3^4}{2^4+1} = (-1) \cdot \frac{81}{16+1} = -\frac{81}{17}$;

$a_5 = (-1)^{5-1} \cdot \frac{3^5}{2^5+1} = 1 \cdot \frac{243}{32+1} = \frac{243}{33} = \frac{81}{11}$.

The sequence is $1, -\frac{9}{5}, 3, -\frac{81}{17}, \frac{81}{11}, \cdots$.

21. $a_{14} = (-1)^{14}[6(14) + 5] = 1 \cdot (84 + 5) = 89$

24. $a_8 = 2(8)^2[3(8) - 1] = 2(64)(23) = 2{,}944$

30. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \cdots$ The numerators are all 1 and the denominators are powers of 3. So $a_n = \frac{1}{3^n}$.

35. $-6, 10, -14, 18, -22, \cdots$ The signs alternate so we have factor $(-1)^n$. Since $10 - 6 = 4$, $14 - 10 = 4$, $18 - 14 = 4$, there is a common difference of 4, so a term of the general term is $4n$. Since $4(1) + 2 = 6$ and $4(2) + 2 = 10$, we find the general term is $a_n = (-1)^n(4n + 2)$. **37.** Since the culture triples every hour, we want **a.** $a_3 = 1{,}000(3^3) = 27{,}000$ **b.** $a_5 = 1{,}000(3^5) = 243{,}000$ **c.** $a_n = 1{,}000(3^n)$.

Review exercises

1. a. $f(-2) = 6$ **b.** $f(0) = 4$ **c.** $f(2) = 14$ **2.** -10 **3.** $\left\{\left(\frac{8}{7}, -\frac{4}{7}\right)\right\}$ **4.** $x = -3$ **5.** $x + 3y = -4$ **6.** $\left\{-\frac{1}{2}\right\}$

Exercise 12–2

Answers to odd-numbered problems

1. $S_5 = 75$ **3.** $S_3 = \frac{47}{20}$ **5.** $S_3 = -33$ **7.** $S_3 = \frac{41}{6}$ **9.** $S_5 = 55$ **11.** $S_6 = 60$ **13.** $S_5 = 10$ **15.** $S_4 = 50$ **17.** $S_5 = 10$ **19.** $S_5 = \frac{743}{840}$ **21.** $S_5 = \frac{937}{168}$ **23.** $S_5 = \frac{5269}{900}$ **25.** $S_4 = \frac{7}{36}$ **27.** $S_3 = 24$ **29.** 45 **31.** $\frac{77}{60}$ **33.** $\frac{573}{60}$ **35.** -13 **37.** $\sum_{i=1}^{5} = i$ **39.** $\sum_{i=1}^{6} i^3$ **41.** $\sum_{i=1}^{4}\left(\frac{i+1}{i+2}\right)$ **43.** $\sum_{i=1}^{4}\left(\frac{2i}{3^{i-1}}\right)$ **45.** $\sum_{i=1}^{4}\left(\frac{2i+3}{3i+1}\right)$ **47.** $\sum_{i=1}^{6}(-1)^i 2^i$

Solutions to trial exercise problems

2. $S_4 = [5(1) - 1] + [5(2) - 1] + [5(3) - 1] + [5(4) - 1]$
$= 4 + 9 + 14 + 19 = 46$

13. $\sum_{k=1}^{5} k(k-3) = 1(1-3) + 2(2-3) + 3(3-3) + 4(4-3) + 5(5-3) = -2 + (-2) + 0 + 4 + 10 = 10$

19. $\sum_{k=1}^{5} \frac{1}{k+3} = \frac{1}{1+3} + \frac{1}{2+3} + \frac{1}{3+3} + \frac{1}{4+3} + \frac{1}{5+3}$
$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$
$= \frac{210 + 168 + 140 + 120 + 105}{840} = \frac{743}{840}$

24. $\sum_{j=1}^{5} (-1)^j \cdot \frac{3}{2j} = (-1)^1 \cdot \frac{3}{2(1)} + (-1)^2 \cdot \frac{3}{2(2)} + (-1)^3 \cdot \frac{3}{2(3)} + (-1)^4 \cdot \frac{3}{2(4)} + (-1)^5 \cdot \frac{3}{2(5)}$
$= \left(-\frac{3}{2}\right) + \frac{3}{4} + \left(-\frac{1}{2}\right) + \frac{3}{8} + \left(-\frac{3}{10}\right)$
$= -\frac{23}{10} + \frac{9}{8} = \frac{-92 + 45}{40} = -\frac{47}{40}$

33. $\sum_{j=0}^{5} \frac{2j+1}{j+1} = \frac{2(0)+1}{0+1} + \frac{2(1)+1}{1+1} + \frac{2(2)+1}{2+1} + \frac{2(3)+1}{3+1} + \frac{2(4)+1}{4+1} + \frac{2(5)+1}{5+1} = 1 + \left(\frac{3}{2}\right) + \left(\frac{5}{3}\right) + \left(\frac{7}{4}\right) + \left(\frac{9}{5}\right) + \left(\frac{11}{6}\right) = \frac{573}{60}$

41. $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$; The numerator is the number of the term plus 1, that is, $(n + 1)$, and the denominator is 2 plus the number of the term. Then $a_n = \frac{n+1}{n+2}$ and we have 4 terms. $\sum_{i=1}^{4}\left(\frac{i+1}{i+2}\right)$

46. $2 - 5 + 8 - 11 + 14$; The signs alternate starting with the first term positive, $(-1)^{n+1}$. Each term differs by 3 and the first term is $3(1) - 1$, second term is $3(2) - 1$. So $a_n = (-1)^{n+1}(3n - 1)$ and we have $\sum_{j=1}^{5}(-1)^{j+1}(3j - 1)$.

Review exercises

1. 11 **2.** 10 **3.** 22 **4.** 54 **5.** $-\frac{1}{18}$ **6.** $\{2.262\}$

Exercise 12–3

Answers to odd-numbered problems

1. arithmetic; $d = 1$ **3.** arithmetic; $d = 2$ **5.** not arithmetic **7.** arithmetic; $d = \frac{1}{2}$ **9.** arithmetic; $d = \frac{5}{3}$ **11.** $a_{16} = 79$ **13.** $a_{17} = -58$ **15.** $a_{12} = \frac{17}{3}$ **17.** $a_{14} = 9$ **19.** $a_{16} = 65$ **21.** $a_{25} = 105$ **23.** $n = 6$ terms **25.** $n = 15$ terms **27.** $n = 29$ terms **29.** $S_{16} = 288$ **31.** $S_{14} = -175$ **33.** $S_{18} = 85\frac{1}{2}$ **35.** $S_{14} = 280$ **37.** $S_{19} = -95$ **39.** $S_{15} = 165$ **41.** $S_{22} = -440$ **43.** $S_{17} = 51$ **45.** $S_{10} = 13$ **47.** $S_{13} = 494$ **49.** $S_{15} = -165$ **51.** $S_{11} = -\frac{319}{6}$ **53.** 121 cans **55.** 112 cans **57.** 3,422 **59.** 240 ft; 1,600 ft **61.** \$1,010 per month **63.** \$82,080 **65.** \$590

Solutions to trial exercise problems

4. Since $6 - 4 = 2$, $8 - 6 = 2$, $10 - 8 = 2$, the sequence is arithmetic and $d = 2$. **7.** Since $2 - \frac{3}{2} = \frac{1}{2}$, $\frac{5}{2} - 2 = \frac{1}{2}$, $3 - \frac{5}{2} = \frac{1}{2}$, the sequence is arithmetic and $d = \frac{1}{2}$.

15. Using $a_n = a_1 + (n-1)d$, $a_{12} = 2 + (12-1)\frac{1}{3}$ $= 2 + (11)\frac{1}{3} = 2 + \frac{11}{3} = \frac{17}{3}$. **28.** Using $a_n = a_1 + (n-1)d$, we want n when $a_n = -2$, $a_1 = \frac{5}{3}$, and $d = \frac{4}{3} - \frac{5}{3} = -\frac{1}{3}$. Then $-2 = \frac{5}{3} + (n-1)\left(-\frac{1}{3}\right)$; $-2 = \frac{5}{3} - \frac{1}{3}n + \frac{1}{3}$; $-2 = 2 + \frac{1}{3}n$; $-4 = -\frac{1}{3}n$; $n = 12$. **33.** Using $S_n = \frac{n}{2}(a_1 + a_n)$, we want S_{18} when $n = 18$, $a_1 = \frac{1}{2}$, and $a_{18} = 9$. So $S_{18} = \frac{18}{2}\left(\frac{1}{2} + 9\right) = 9\left(\frac{19}{2}\right) = \frac{171}{2}$ or $85\frac{1}{2}$. **37.** Using $S_n = \frac{n}{2}(a_1 + a_n)$, we want S_{19} when $n = 19$, $a_1 = 5 - 1 = 4$; $a_{19} = 5 - 19 = -14$. $S_{19} = \frac{19}{2}[4 + (-14)] = \frac{19}{2}(-10)$ $= -\frac{190}{2} = -95$. **44.** $\sum_{k=1}^{14} \frac{1}{2}k = \frac{14}{2}\left(\frac{1}{2} + 7\right) = 7\left(\frac{15}{2}\right)$ $= \frac{105}{2}$ or $52\frac{1}{2}$. **51.** We want S_{11} using $S_n = \frac{n}{2}(a_1 + a_n)$. When $n = 11$, $a_1 = \frac{1}{6}$, and $a_{11} = \frac{1}{6} + (11-1)(-1) = \frac{1}{6}$ $+ (10)(-1) = \frac{1}{6} + (-10) = -\frac{59}{6}$. So $S_{11} = \frac{11}{2}\left[\frac{1}{6} + \left(-\frac{59}{6}\right)\right]$ $= \frac{11}{2}\left(-\frac{58}{6}\right) = \frac{11}{2}\left(-\frac{29}{3}\right) = -\frac{319}{6}$ or $-53\frac{1}{6}$. **54.** We want S_n when $a_1 = 30$, $d = 27 - 30 = -3$, and $a_n = 3$. Now, using $a_n = a_1 + (n-1)d$, we have $3 = 30 + (n-1)(-3)$; -27 $= -3n + 3$; $-30 = -3n$; $n = 10$. So $S_{10} = \frac{10}{2}(30 + 3)$ $= 5(33) = 165$ boxes. **62.** We want n when $S_n = 3{,}600$, $a_1 = 16$, and $d = 32$. Using $S_n = \frac{n}{2}[2a_1 + (n-1)d]$, $3{,}600 = \frac{n}{2}[2(16) + (n-1)\cdot 32]$; $3{,}600 = \frac{n}{2}[32 + 32n - 32]$; $3{,}600 = \frac{n}{2}(32n)$; $3{,}600 = 16n^2$; $n^2 = 225$; $n = 15$ sec.

65. We want a_{19} when $a_1 = 50$, $n = 19$, and $d = 30$. Using $a_n = a_1 + (n-1)d$, $a_{19} = 50 + (19-1)30 = 50 + 18(30) = 50 + 540 = 590$. Thus \$590 was deposited on her eighteenth birthday.

Review exercises

1. $\frac{5}{y-6}$ **2.** 5 **3.** $\{4\}$ **4. a.** parabola **b.** hyperbola **c.** ellipse **5.** $-2 - \sqrt{6}$ **6.** $\frac{3(a+1)}{3a+5}$

Exercise 12-4

Answers to odd-numbered problems

1. geometric; 27,81,243; $r = 3$ **3.** geometric; $\frac{1}{54}, \frac{1}{162}, \frac{1}{486}$; $r = \frac{1}{3}$ **5.** geometric; $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$; $r = -\frac{1}{2}$ **7.** geometric; $\frac{-2}{9}, \frac{2}{27}, \frac{-2}{81}$; $r = -\frac{1}{3}$ **9.** not geometric **11.** $a_n = 3(2)^{n-1}$ **13.** $a_n = 27\left(\frac{-2}{3}\right)^{n-1}$ **15.** $a_n = (\sqrt{3})^{n-1}$ **17.** $a_n = \frac{-1}{15}(-3)^{n-1}$ **19.** $a_5 = 162$ **21.** $a_4 = 3$ **23.** $a_6 = -160$ **25.** $a_5 = -\frac{1}{8}$ **27.** $a_7 = 576$ **29.** $a_9 = 1{,}792$ **31.** $S_5 = 1{,}694$ **33.** $S_4 = -\frac{312}{25}$ **35.** $S_6 = 27{,}993$ **37.** $S_9 = \frac{511}{512}$ **39.** $S_5 = -305$ **41.** $S_8 = 87{,}380$ **43.** $S_5 = -183$ **45.** $S_7 = \frac{4{,}118}{2{,}187}$ **47.** $S_6 = \frac{31{,}122}{15{,}625}$ **49.** $S_6 = -\frac{22{,}344}{3{,}125}$ **51.** $40\frac{7}{27}$ ft **53.** \$512 **55.** \$10,737,418.24 **57.** $\frac{2{,}101}{3{,}125} \approx 0.67$ of the tank

Solutions to trial exercise problems

5. Since $\frac{-2}{4} = -\frac{1}{2}$ and $\frac{1}{-2} = -\frac{1}{2}$ the sequence is geometric with $r = -\frac{1}{2}$ and the next three terms are $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$.

15. Using $a_n = a_1r^{n-1}$, since $\frac{\sqrt{3}}{1} = \sqrt{3}$ and $\frac{3}{\sqrt{3}} = \sqrt{3}$, then $r = \sqrt{3}$ and $a_1 = 1$. Thus $a_n = 1(\sqrt{3})^{n-1} = (\sqrt{3})^{n-1}$.

25. Since $a_1 = -32$ and $r = -\frac{1}{4}$, using $a_n = a_1r^{n-1}$, $a_5 = (-32)\left(-\frac{1}{4}\right)^{5-1} = (-32)\left(-\frac{1}{4}\right)^4 = (-32)\left(\frac{1}{256}\right)$ $= -\frac{1}{8}$. **26.** $r = \frac{18}{3} = 6$ and $a_1 = 3$, then $a_6 = 3(6)^{6-1}$ $= 3(6)^5 = 3(7{,}776) = 23{,}328$. **33.** Using $S_n = \frac{a_1 - a_1r^n}{1-r}$, given $a_1 = -10$ and $r = \frac{1}{5}$, $S_4 = \dfrac{-10 - (-10)\left(\frac{1}{5}\right)^4}{1 - \frac{1}{5}}$

$$= \frac{-10 + 10\left(\frac{1}{625}\right)}{\frac{4}{5}} = \frac{-10 + \frac{2}{125}}{\frac{4}{5}} = \frac{-1{,}250 + 2}{100}$$

$= \frac{-1{,}248}{100} = -12\frac{12}{25}$. **34.** $r = \frac{18}{9} = 2$ and $a_1 = 9$, so $S_7 = \frac{9 - 9(2)^7}{1-2} = \frac{9 - 9(128)}{-1} = \frac{9 - 1{,}152}{-1} = 1{,}143$.

42. $r = -2$ and $a_1 = -2$. We want S_7.

$$S_7 = \frac{-2 - (-2)(-2)^7}{1 - (-2)} = \frac{-2 + 2(-2)^7}{3} = \frac{-2 + 2(-128)}{3} = \frac{-2 - 256}{3} = \frac{-258}{3} = -86$$

47. $r = \frac{2}{5}$ and $a_1 = 3\left(\frac{2}{5}\right) = \frac{6}{5}$. We want $S_6 \cdot S_6 = \dfrac{\frac{6}{5} - \frac{6}{5}\left(\frac{2}{5}\right)^6}{1 - \frac{2}{5}}$

$$= \frac{\frac{6}{5} - \frac{6}{5}\left(\frac{64}{15{,}625}\right)}{\frac{3}{5}} = \frac{6 - \frac{384}{15{,}625}}{3} = 2 - \frac{128}{15{,}625} = \frac{31{,}122}{15{,}625}.$$

51. Since the ball will travel each height *twice*, after the initial drop of 9 ft, then we want $9 + 2\sum_{i=1}^{5} 9\left(\frac{2}{3}\right)^i$. Now

$$2\sum_{i=1}^{5} 9\left(\frac{2}{3}\right)^i = 2\left[\frac{6 - 6\left(\frac{2}{3}\right)^5}{1 - \frac{2}{3}}\right] = 2\left[\frac{6 - 6\left(\frac{2}{3}\right)^5}{\frac{1}{3}}\right]$$

$$= 2\left[\frac{6 - 6\left(\frac{32}{243}\right)}{\frac{1}{3}}\right] = 2\left[\left(6 - \frac{192}{243}\right)\cdot 3\right] = 2\left[18 - \frac{192}{81}\right]$$

$$= 2\left[\frac{1{,}458 - 192}{81}\right] = 2\left(\frac{1{,}266}{81}\right) = 2\left(\frac{422}{27}\right) = \frac{844}{27} \text{ or } 31\frac{7}{27}.$$

The ball has traveled $9 + 31\frac{7}{27} = 40\frac{7}{27}$ ft at the sixth strike.

Review exercises

1. $16x^2 + 8xy + y^2$ **2.** $27x^3 - 54x^2y + 36xy^2 - 8y^3$
3. $x^2 - 12x + 36 = (x - 6)^2$ **4.** $\frac{4}{3}$ **5.** $\frac{6}{5} + \frac{3}{5}i$
6. $\left\{\frac{1 - i\sqrt{31}}{4}, \frac{1 + i\sqrt{31}}{4}\right\}$

Exercise 12-5

Answers to odd-numbered problems

1. 3 **3.** -6 **5.** $\frac{9}{10}$ **7.** $-\frac{1}{2}$ **9.** 18 **11.** 5
13. no sum **15.** 4 **17.** $-\frac{2}{5}$ **19.** $\frac{3}{4}$ **21.** $\frac{1}{3}$ **23.** $\frac{31}{110}$
25. $\frac{2}{55}$ **27.** 570 in. **29.** 84 in. **31.** 25 mg

Solutions to trial exercise problems

3. Using $S_\infty = \frac{a_1}{1 - r}$, $S_\infty = \frac{-3}{1 - \frac{1}{2}} = \frac{-3}{\frac{1}{2}} = -6$.

6. Using $S_\infty = \frac{a_1}{1 - r}$, $S_\infty = \frac{4}{1 - \left(-\frac{1}{2}\right)} = \frac{4}{\frac{3}{2}} = 4 \cdot \frac{2}{3}$ $= \frac{8}{3}$. **9.** $a_1 = 12$ and $r = \frac{4}{12} = \frac{1}{3}$, so $S_\infty = \frac{12}{1 - \frac{1}{3}}$ $= \frac{12}{\frac{2}{3}} = 12 \cdot \frac{3}{2} = 18$. **13.** $a_1 = 6$ and $r = \frac{-8}{6} = -\frac{4}{3}$

So S_∞ does not exist since $\left|-\frac{4}{3}\right| > 1$. **16.** Now $a_1 = \left(\frac{7}{8}\right)^{1+1}$ $= \left(\frac{7}{8}\right)^2 = \frac{49}{64}$ and $r = \frac{7}{8}$, so $\sum_{k=1}^{\infty}\left(\frac{7}{8}\right)^{k+1} = \frac{\frac{49}{64}}{1 - \frac{7}{8}}$

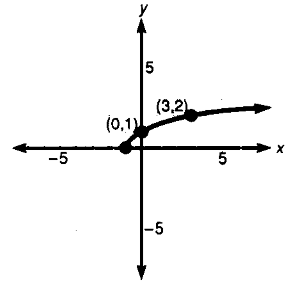
$$= \frac{\frac{49}{64}}{\frac{1}{8}} = \frac{49}{64}\cdot\frac{8}{1} = \frac{49}{8} \text{ or } 6\frac{1}{8}.$$

23. $0.28181\overline{81} = 0.2 + 0.08181\overline{81}$. Now for $0.08181\overline{81}$, $a_1 = 0.081$ and $r = \frac{0.00081}{0.081} = 0.01$. Then $0.08181\overline{81} = \frac{0.081}{1 - 0.01} = \frac{0.081}{0.99}$ $= \frac{81}{990} = \frac{9}{110}$. Thus $0.28181\overline{81} = \frac{2}{10} + \frac{9}{110} = \frac{22 + 9}{110} = \frac{31}{110}$.

28. Now $a_1 = 16$ cm and $r = \frac{7}{8}$. We want $S_\infty = \frac{a_1}{1 - r}$ $= \frac{16}{1 - \frac{7}{8}} = \frac{16}{\frac{1}{8}} = 16 \cdot 8 = 128$. The bob travels 128 cm before coming to rest.

Review exercises

1. $x^3 + 3x^2y + 3xy^2 + y^3$ **2.** $\frac{1}{a^2}$ **3.** a^3b^2
4. $f(x) = \sqrt{x + 1}$



5. $f[g(x)] = 4x^2$ **6.** $(a + b)(a - b)(a^2 + b^2)$
7. $3(x + 3y)(x - 3y)$ **8.** $3(x + 2y)(x^2 - 2xy + 4y^2)$

Exercise 12-6

Answers to odd-numbered problems

1. 720 **3.** 11,880 **5.** 10 **7.** 28 **9.** 120
11. $a^4 - 12a^3 + 54a^2 - 108a + 81$
13. $p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$
15. $16a^4 + 96a^3 + 216a^2 + 216a + 81$
17. $\frac{p^6}{64} - \frac{3}{16}p^5q + \frac{15}{16}p^4q^2 - \frac{5}{2}p^3q^3 + \frac{15}{4}p^2q^4 - 3pq^5 + q^6$
19. $a^{10} + 5a^8b^2 + 10a^6b^4 + 10a^4b^6 + 5a^2b^8 + b^{10}$
21. $-2{,}002a^9b^5$ **23.** $7{,}920q^8$ **25.** $18{,}144\,k^6$ **27.** 1.0263
29. .8587 **31.** $924a^9 (a > 0)$ **33.** 70

Solutions to trial exercise problems

5. $\frac{10!}{9!} = \frac{10\cdot 9!}{9!} = 10$ **11.** $(a - 3)^4 = a^4 + \frac{4}{1!}a^3(-3)^1$ $+ \frac{4\cdot 3}{2!}a^2(-3)^2 + \frac{4\cdot 3\cdot 2}{3!}a(-3)^3 + \frac{4\cdot 3\cdot 2\cdot 1}{4!}(-3)^4$ $= a^4 - 12a^3 + 54a^2 - 108a + 81$ **19.** $(a^2 + b^2)^5$

$$= (a^2)^5 + \frac{5}{1!}(a^2)^4(b^2)^1 + \frac{5\cdot 4}{2!}(a^2)^3(b^2)^2 + \frac{5\cdot 4\cdot 3}{3!}(a^2)^2(b^2)^3$$
$$+ \frac{5\cdot 4\cdot 3\cdot 2}{4!}(a^2)(b^2)^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{5!}(b^2)^5$$
$$= a^{10} + 5a^8b^2 + 10a^6b^4 + 10a^4b^6 + 5a^2b^8 + b^{10}$$

25. Given $(6-k)^9$, we want the 7th term, where $n = 9$, $r = 7$, $x = 6$, and $y = -k$. Using $\frac{n!}{[n-(r-1)]!(r-1)!}$,
$x^{n-(r-1)}y^{r-1} = \frac{9!}{3!6!}(6)^3(-k)^6 = \frac{9\cdot 8\cdot 7}{3!}(216)(k)^6 = 84(216)k^6$
$= 18{,}144k^6$. **27.** We want $(1.002)^{13} = (1+0.002)^{13}$
$= 1^{13} + \frac{13}{1!}(1)^{12}(0.002) + \frac{13\cdot 12}{2!}(1)^{11}(0.002)^2$
$+ \frac{13\cdot 12\cdot 11}{3!}(1)^{10}(0.002)^3 = 1 + 0.026 + 0.000312 + \cdots$
$= 1.026312 \approx 1.0263$. **31.** We want the 7th term of $(a+\sqrt{a})^{12}$. Now $n = 12$, $r = 7$, $x = a$, and $y = \sqrt{a}$.
So, $\frac{n!}{[n-(r-1)]!(r-1)!}x^{n-(r-1)}y^{(r-1)} = \frac{12!}{6!6!}a^6(\sqrt{a})^6$
$= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}(a^6)(a)^3 = 11\cdot 3\cdot 4\cdot 7 \quad a^9 = 924a^9$.
The middle term of $(a+\sqrt{a})^{12}$ is $924a^9$. $(a > 0)$

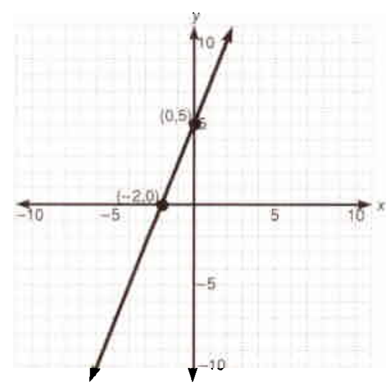
Chapter 12 review

1. 7,11,15,19,23 **2.** $5, \frac{10}{3}, 3, \frac{20}{7}, \frac{25}{9}$ **3.** $\frac{-4}{7}, \frac{4}{9}, \frac{-4}{11}, \frac{4}{13}, \frac{-4}{15}$
4. 2,−4,8,−16,32 **5.** $a_6 = -14$ **6.** $a_7 = -17$
7. $a_9 = \frac{6561}{2}$ **8.** $a_{11} = \frac{683}{59{,}049}$ **9.** $a_n = 2n+3$
10. $a_n = 5n - 2$ **11.** $a_n = \frac{n+1}{4n-1}$ **12.** $a_n = (-1)^n(5n-1)$
13. $a_n = 0.25n + 2.75$; $a_8 = \$4.75$ **14.** $S_4 = 36$
15. $S_6 = 196$ **16.** $S_5 = \frac{229}{20}$ **17.** $S_6 = -\frac{37}{75}$
18. $\sum_{i=1}^{4}(3i+2)$ **19.** $\sum_{k=1}^{5}(-1)^{k+1}\left(\frac{k+3}{k+4}\right)$ **20.** $a_{15} = 61$
21. $a_{17} = 77$ **22.** $a_{21} = 83$ **23.** $a_{19} = -42$ **24.** $n = 15$
25. $n = 21$ **26.** $S_{15} = 585$ **27.** $S_{21} = -420$
28. $S_{29} = 290$ **29.** $S_{25} = \frac{375}{2}$ **30.** \$16,500
31. $a_n = 3(2)^{n-1}$ **32.** $a_n = \left(-\frac{3}{4}\right)^n$ **33.** $a_5 = 405$
34. $a_4 = -\frac{8}{9}$ **35.** $a_3 = 16$ **36.** $a_7 = -576$
37. $S_5 = 363$ **38.** $S_6 = -\frac{189}{4}$ **39.** $S_5 = -\frac{543}{1{,}024}$
40. $S_7 = \frac{4{,}372}{2{,}187}$ **41.** $S_\infty = 12$ **42.** $S_\infty = -\frac{5}{2}$
43. $S_\infty = \frac{-3}{5}$ **44.** $S_\infty = -\frac{1}{45}$ **45.** $\frac{35}{99}$ **46.** $\frac{214}{495}$
47. 24 meters **48.** $x^7 + 35x^6 + 525x^5 + 4375x^4 + 21{,}875x^3 + 65{,}625x^2 + 109{,}375x + 78{,}125$ **49.** $32a^5 - 240a^4b + 720a^3b^2 - 1080a^2b^3 + 810ab^4 - 243b^5$
50. $\frac{a^4}{16} - \frac{3a^3b}{2} + \frac{27a^2b^2}{2} - 54ab^3 + 81b^4$ **51.** $84{,}480a^7$
52. $19{,}702{,}683a^8b^6$ **53.** $608{,}256x^{10}y^2$ **54.** $\frac{27}{32}$ ft. **55.** 1,287

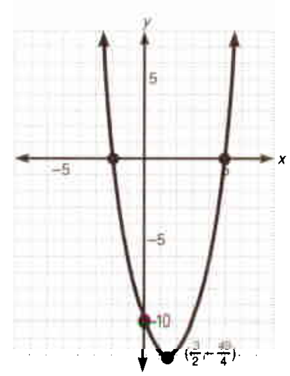
Final examination

1. $\{-7,-6,-5,-4,-3,-2,-1,0,1,2,3\}$ **2. a.** $\{2,7\}$
b. $\{-4,-1,0,2,4,7,9\}$ **c.** $\emptyset$ **3.** $\{-4,-3,-2,-1,0,1,2,3,4,5,6,7,8\}$
4. 13 **5. a.** **b.** $\frac{a^6}{4b^4}$ **c.** $\frac{b^4}{a^6}$ **6. a.** $4y^2 + 28y + 49$
b. $16x^2 - 4y^2$ **c.** $x^3 - 8$ **7.** $4xy(3 - xy^2 + 2x^2y)$
8. $(7y+1)(y-5)$ **9.** $(2a-5b)^2$ **10.** $2(2x+5y)(2x-5y)$
11. $2(2a-b)(4a^2+2ab+b^2)$ **12.** $(3a-b)(2x-y)$
13. $\left\{\frac{21}{25}\right\}$ **14.** $\left\{x \middle| x < -\frac{29}{2}\right\} = \left(-\infty, -\frac{29}{2}\right)$
15. $\left\{x \middle| -\frac{5}{2} < x \le 1\right\} = \left(-\frac{5}{2}, 1\right]$ **16.** $\{4,1\}$
17. $\left\{x \middle| \frac{1}{3} < x < 3\right\} = \left(\frac{1}{3}, 3\right)$ **18.** $\left\{x \middle| x \le -\frac{2}{5} \text{ or } x \ge 2\right\}$
$= \left(-\infty, -\frac{2}{5}\right] \cup [2,\infty)$ **19.** $\{8,-3\}$ **20.** $\left\{-\frac{5}{4}\right\}$
21. $\frac{32a+9}{(2a-1)(4a+3)}$ **22.** $\frac{y^2-5y+1}{(y-7)(y+6)(y-6)}$
23. $\frac{3a^2-4a+1}{a^2-3a-4}$ **24.** $\frac{2(2b-a)}{2a-b}$ **25.** $3x^2 + 13x + 37 + \frac{112}{x-3}$
26. $17\sqrt{3}$ **27.** $\frac{4+7i}{13}$ or $\frac{4}{13} + \frac{7}{13}i$
28. $12 - 8\sqrt{5} + 9\sqrt{2} - 6\sqrt{10}$ **29.** $\left\{\frac{7+\sqrt{73}}{6}, \frac{7-\sqrt{73}}{6}\right\}$
30. $\{5\}$, 2 is extraneous **31.** $\{y \mid 1 \le y \le 3\} = [1,3]$
32. a. $3x - 2y = -1$ **b.** $x + 4y = -8$ **c.** $2x + 3y = 0$
33. $m = \frac{3}{5}$, $b = \frac{9}{5}$ **34. a.** −18 **b.** 13 **c.** 5

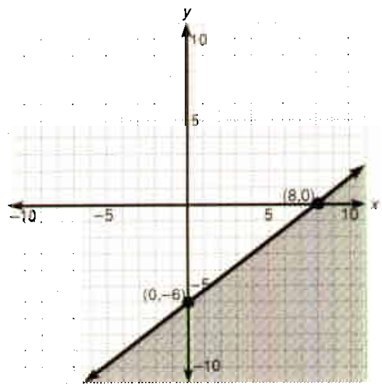
35.



36.



37.



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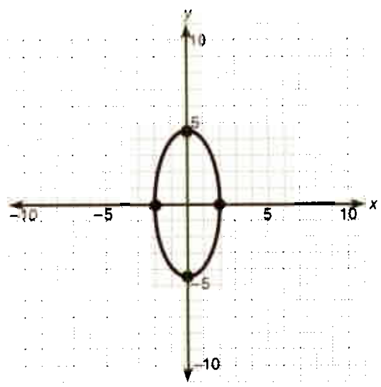

39. a. circle **b.** hyperbola **c.** parabola **d.** ellipse

40. $\left\{\left(\frac{11}{19}, -\frac{12}{19}\right)\right\}$ **41.** 149 **42.** $\left\{\left(\frac{29}{11}, -\frac{5}{11}\right)\right\}$ **43.** $\log_b\left(\frac{15}{4}\right)$

44. 1.7712 **45.** 0.738 **46.** 3.5835 **47.** $\sum_{i=1}^{5}(6i - 3)$

48. 49 **49.** -44 **50.** $\frac{351}{2}$ **51.** $-\frac{1}{54}$ **52.** $-\frac{2}{3}$

53. $81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4$ **54.** $14{,}437{,}500a^6$

55. $\frac{26}{111}$ **56.** 462

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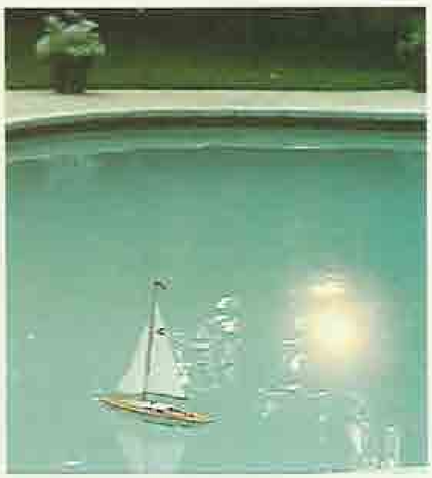

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